Performance Bounds of Model Predictive Control for Unconstrained and Constrained Linear Quadratic Problems and Beyond

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- The framework developed in [Ber22] couches on the dynamic programming (DP) theory, and regards MPC as an approximate scheme for solving a functional equation.
- We leverage on the new framework, and investigate how the terminal ingredients affect the performance of MPC.

Difference in Perspective

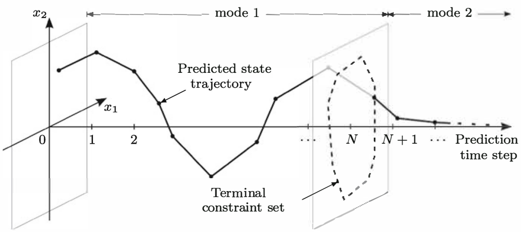

Figure source: [KC16, Fig. 2.1]

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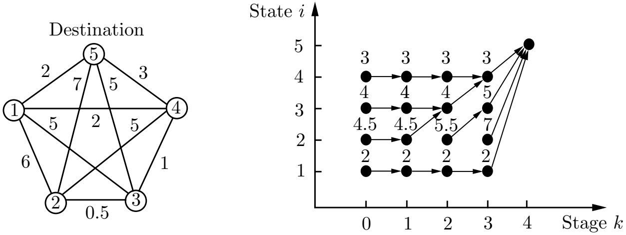

(a) (b)

Figure source: [Ber17, Fig. 2.1.2]

- MPC analyzes the predicted trajectory into the future: Forward in time, focusing on trajectories of states.
- DP analysis proceeds as the algorithm progress: Forward in algorithmic iteration, focusing on functions of states.

Dynamic Programming Model

The scalar linear quadratic regulation (LQR) problem:

$$x_{k+1} = ax_k + bu_k, \quad \min_{u_k, k=0,1,\dots} \lim_{N\to\infty} \sum_{k=0}^{N-1} (qx_k^2 + ru_k^2).$$

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- The **system dynamics** $f : X \times U \to X$ and the **stage cost** $g : X \times U \to \Re$

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- A **policy** $\mu : X \to U$ with $\mu(x) \in U(x)$ for all x and its **cost function**

$$\mu(x) = Lx, \quad J_\mu(x_0) = \lim_{N\to\infty} \sum_{k=0}^{N-1} \left(qx_k^2 + r(Lx_k)^2\right).$$

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- The **optimal cost function** $J^* : X \to \Re$ and **optimal policy** $\mu^* : X \to U$:

$$J^*(x_0) = \min_{u_k, k=0,1,\ldots} \lim_{N\to\infty} \sum_{k=0}^{N-1} (qx_k^2 + ru_k^2), \quad \mu^*(x) = L^* x.$$

Bellman's Equation

- The optimal cost function J^* fulfills **Bellman's equation**:

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- Bellman's equation holds for a given policy μ:

$$J_\mu(x) = g\big(x, \mu(x)\big) + J_\mu\Big(f\big(x, \mu(x)\big)\Big), \quad \text{for all } x.$$

Value Iteration

- The VI algorithm generates a sequence of functions $\{J_k\}$ by

$$J_{k+1}(x) = \min_{u \in U(x)} \Big[g(x,u) + J_k\big(f(x,u)\big) \Big], \quad k = 0, 1, \ldots .$$

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- **Principle of optimality** yields

$$\begin{aligned}\min_{u_0, u_1 \in \Re} \left[qx_0^2 + ru_0^2 + qx_1^2 + ru_1^2 + Kx_2^2\right] &= \min_{u_0 \in \Re} \left[qx_0^2 + ru_0^2 + \min_{u_1 \in \Re} \left[qx_1^2 + ru_1^2 + Kx_2^2\right]\right] \\ &= \min_{u_0 \in \Re} \left[qx_0^2 + ru_0^2 + F(K)x_1^2\right]\end{aligned}$$

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- The PI algorithm generates a sequence of functions $\{J_{\mu^k}\}$ by **policy evaluation**

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- The policy improvement fulfills

$$(a + bL_{k+1})^2 = \frac{\partial F}{\partial K}(K_k)$$

A Dynamic Programming View: Unconstrained Problem

- Consider MPC scheme for solving LQR:

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$$K_{\tilde{L}} = q + r\tilde{L}^2 + K_{\tilde{L}}(a + b\tilde{L})^2.$$

- The cost $K_{\tilde{L}}$ is the approximate solution to Bellman's equation computed via MPC.

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  with suitably designed K and S.
- Identical interpretation: $\ell - 1$ steps VIs followed by one step policy improvement, with K and S collectively as initial guess.

Main Result

- Proposition 13 (informal): Let $\{x_i^*\}_{i=0}^{\ell}$ be the sequence of states under μ^* starting from x. Then we have

$$J_{\tilde{\mu}}(x) - J^*(x) \leq K(x_\ell^*)^2 - J^*(x_\ell^*) \leq (K - K^*)(x_\ell^*)^2.$$

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- Note that for this problem, μ^* is *not* linear, and J^* is *not* quadratic.
- Given that μ^* drives the system to zero exponentially, even large $K - K^*$ would results in a small bound value, as x_ℓ^* is likely to be small.

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where K^* solves the Riccati equation, and $S^* \subset \hat{X}$ has the property that

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  where L^* is optimal LQR control.
- The reason for choosing $K^* x^2$ as terminal cost is to have good performance $J_{\tilde{\mu}}$, **which seems not necessary, according to our bound!**

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  where a choice for K could be the solution K_L that solves $K_L = q + rL^2 + K_L(a + bL)^2$.
- Since S is larger, the feasible set of initial states x for the modified MPC is also larger, while the performance should be rather comparable.

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$$K = A'\big(K - KB(B'KB + \hat{R})^{-1}B'K\big)A + Q,$$

where $\hat{R} = 50R$.

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Two Dimensional Example

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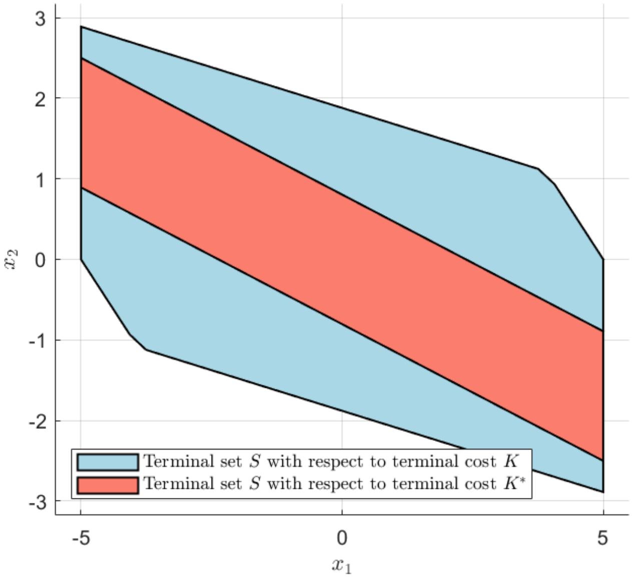
- The set $S \subset \hat{X}$ has the property that

$$x \in S \implies Lx \in U \text{ and } (a + bL)x \in S,$$

  where L is computed as

$$L = -(B'KB + \hat{R})^{-1}B'KA.$$

Two Dimensional Example



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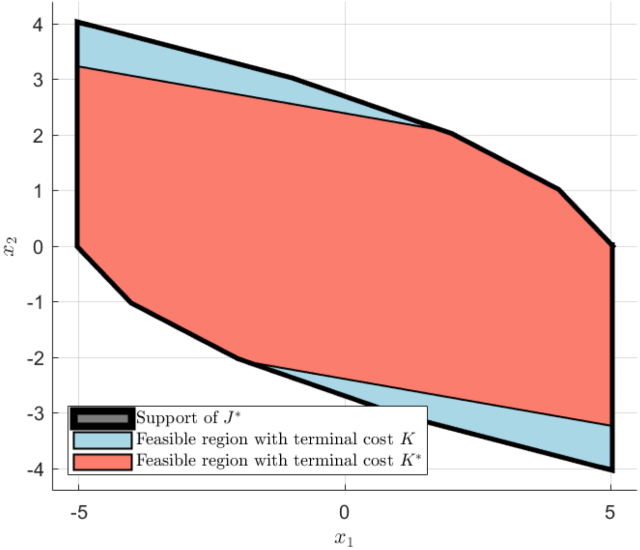

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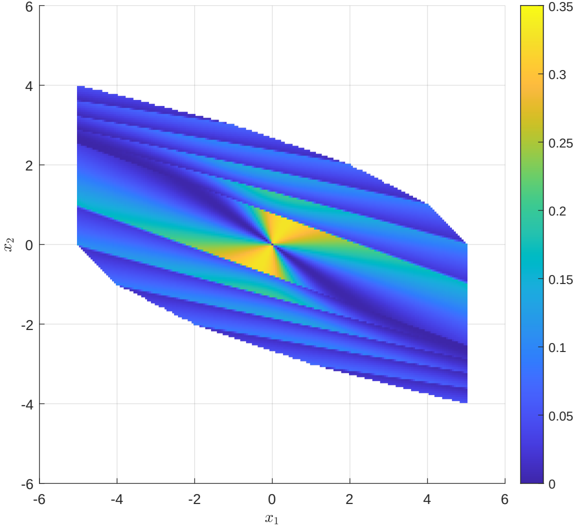

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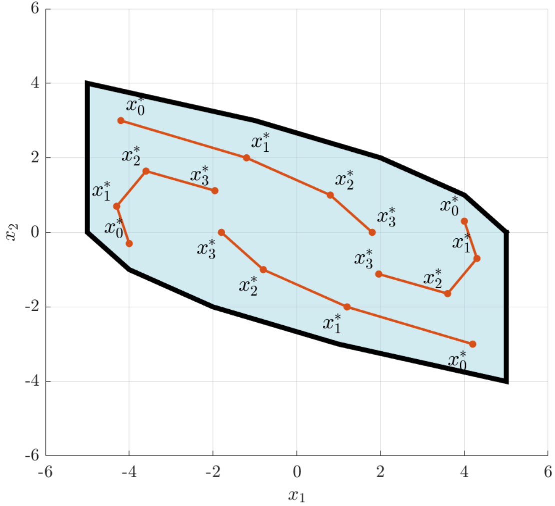

Conclusions

- We analyzed the performance of MPC via the perspective of DP for unconstrained and constrained LQR problems;

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- We analyzed the performance of MPC via the perspective of DP for unconstrained and constrained LQR problems;
- The insights obtained led to new designs of terminal ingredients with larger feasible regions while costing little in performance.

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