KTH, SCHOOL OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

On the Proof of the Law of the Unconscious Statistician

Yuchao Li

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1 PROBLEM STATEMENT

Probability space is given as $(\Omega, \mathscr{F}, \mathbf{P})$ with elementary outcome ω. A random variable (r.v.) defined on the probability space is X. Then with a Borel function $g : \mathbf{R} \to \mathbf{R}$, we know $g(X)$ is $\mathscr{F}_X$-measurable and therefore a r.v. as well. The Law of the Unconscious Statistician (LOTUS) gives the following result

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) dF_X(x). \tag{1.1}$$

By [Hand Note 2], we know the left-hand side is defined as Lebesgue integral while the right-hand side is defined as Lebesgue-Stieltje integral. Prove Eq. (1.1).

2 ELABORATION

Denote $Y = g(X)$, then obviously we have $E[Y] = E[g(X)]$. We prove the cases when $g(x) \geq 0$, namely $Y \geq 0$. P.317 [1] provides the outline of the proof, here we fill in more details.

1. First, when g is simple, Y would be a simple r.v., [2] provides the proof for this case, namely for any simple g_m, the following equation

$$E[g_m\big(X(\omega)\big)] = \int_{-\infty}^{\infty} g_m(x) dF_X(x) \text{ (Lebesgue-Stieltjes)} \tag{2.1}$$

holds. Notice that by saying g_m is simple, we mean that there are Borel sets $G_1, G_2, \ldots, G_m$ that are a partition of $\mathbf{R}$, namely $\cup_{k=1}^m G_m = \mathbf{R}$ such that

$$g_m(x) = g_k, \text{ if } x \in G_k \text{ for } k = 1, 2, \ldots, m. \tag{2.2}$$

2. For some non negative g, since $Y = g(X)$ is a r.v., then its expectation is defined as a Lebesgue integral, refer to [Hand Note 2] and [2, 3, 4] for definition. Then to proof LOTUS is to show that

$$\text{(Lebesgue)} \int_{\Omega} Y(\omega) d\mathbf{P}(\omega) = \int_{-\infty}^{\infty} g(x) dF_X(x) \text{ (Lebesgue-Stieltjes)} \tag{2.3}$$

provided that $Y(\omega) = g\big(X(\omega)\big)$. Again, the left hand side is by definition given as

$$\text{(Lebesgue)} \int_{\Omega} Y(\omega) d\mathbf{P}(\omega) = \lim_{n\to\infty} E[Y_n(\omega)] \tag{2.4}$$

where Y_n is given as

$$Y_n(\omega) = \begin{cases} \frac{k-1}{2^n}, & \frac{k-1}{2^n} \le Y(\omega) < \frac{k}{2^n}, \ k = 1, 2, ..., n2^n \\ n, & Y(\omega) \ge n. \end{cases} \tag{2.5}$$

Plugging in $Y(\omega) = g\big(X(\omega)\big)$, we have

$$Y_n(\omega) = \begin{cases} \frac{k-1}{2^n}, & \frac{k-1}{2^n} \le g\big(X(\omega)\big) < \frac{k}{2^n}, \ k = 1, 2, ..., n2^n \\ n, & g\big(X(\omega)\big) \ge n. \end{cases} \tag{2.6}$$

Denote $f_n : \mathbf{R}_+ \cup \{0\} \to \mathbf{R}$ as

$$f_n(z) = \begin{cases} \frac{k-1}{2^n}, & \frac{k-1}{2^n} \le z < \frac{k}{2^n}, \ k = 1, 2, ..., n2^n, \\ n, & z \ge n. \end{cases} \tag{2.7}$$

We know f_n is Borel function for any fixed n. Then we define g_n as

$$g_n(x) = f_n\big(g(x)\big). \tag{2.8}$$

It can be shown that due to f_n being simple and g Borel, g_n is a simple function, fulfilling the condition given in Eq. (2.2). What's more, $g_n(x) \uparrow g(x)$ pointwise. Then, we have

$$g_n\big(X(\omega)\big) = \begin{cases} \frac{k-1}{2^n}, & \frac{k-1}{2^n} \le g\big(X(\omega)\big) < \frac{k}{2^n}, \ k = 1, 2, ..., n2^n \\ n, & g\big(X(\omega)\big) \ge n. \end{cases} \tag{2.9}$$

Compare Eqs. (2.6) and (2.9), we see that $Y_n(\omega) = g_n\big(X(\omega)\big)$. Substituting $Y_n(\omega)$ with $g_n\big(X(\omega)\big)$ in Eq. (2.4), we have

$$\text{(Lebesgue)} \int_{\Omega} Y(\omega) d\mathbf{P}(\omega) = \lim_{n\to\infty} E[g_n\big(X(\omega)\big)]. \tag{2.10}$$

The first part has shown for any simple g_n, we have $E[g_n\big(X(\omega)\big)] = \int_{-\infty}^{\infty} g_n(x) dF_X(x)$, plug it into Eq. (2.10), we have

$$\text{(Lebesgue)} \int_{\Omega} Y(\omega) d\mathbf{P}(\omega) = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x) dF_X(x) \tag{2.11}$$

where for each n, $\int_{-\infty}^{\infty} g_n(x) dF_X(x)$ is Lebesgue-Stieltjes integral.

3. Recall the definition of Lebesgue-Stieltjes integral, given as Eq. (2.9) of [Hand Note 2], which is

$$\text{(Lebesgue-Stieltjes)} \int_{-\infty}^{\infty} g(x)dF_X(x) = \int_{-\infty}^{\infty} g\big(Y(r)\big)d\tilde{\mathbf{P}}(r) \text{ (Lebesgue).} \tag{2.12}$$

where $Y(r) = r$ is the canoical r.v. of X defined on $(\mathbf{R}, \mathscr{B})$, $\tilde{\mathbf{P}}$ is the probability measure of the probability space $(\mathbf{R}, \mathscr{B}, \tilde{\mathbf{P}})$, which is induced by the canoical r.v. of X, refer to Steps 3 and 4 of [Hand Note 2] for more details. Since for each n, $\int_{-\infty}^{\infty} g_n(x)dF_X(x)$ is Lebesgue-Stieltjes integral, then by Eq. (2.12), we have

$$\text{(Lebesgue-Stieltjes)} \int_{-\infty}^{\infty} g_n(x)dF_X(x) = \int_{-\infty}^{\infty} g_n(r)d\tilde{\mathbf{P}}(r) \text{ (Lebesgue).} \tag{2.13}$$

Similarly, for g, we have

$$\text{(Lebesgue-Stieltjes)} \int_{-\infty}^{\infty} g(x)dF_X(x) = \int_{-\infty}^{\infty} g(r)d\tilde{\mathbf{P}}(r) \text{ (Lebesgue).} \tag{2.14}$$

Recall that $g_n \uparrow g$ point-wise and g_n is defined by Eq. (2.8), due to Lemma 2.1, P. 34, and Definition 2.5 (Lebegue integral), P. 35 in [3], we know by definition, the Lebesgue integral of a non-simple g is given as

$$\text{(Lebesgue)} \int_{-\infty}^{\infty} g(r)d\tilde{\mathbf{P}}(r) = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(r)d\tilde{\mathbf{P}}(r) \tag{2.15}$$

where the right-hand side is limit of Lebesgue integrals. Due to Eq. (2.13), we have

$$\text{(Lebesgue)} \int_{-\infty}^{\infty} g(r)d\tilde{\mathbf{P}}(r) = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)dF_X(x) \tag{2.16}$$

where the right-hand side now is the limit of Lebesgue-Stieltjes integrals.

Substitute $\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)dF_X(x)$ due to Eq. (2.14) and substitute $\int_{-\infty}^{\infty} g(r)d\tilde{\mathbf{P}}(r)$ due to Eq. (2.11), and we can rewrite Eq. (2.16) as

$$\text{(Lebesgue)} \int_{\Omega} Y(\omega)d\mathbf{P}(\omega) = \int_{-\infty}^{\infty} g_n(x)dF_X(x) \text{ (Lebesgue-Stieltjes)}$$

which is our goal Eq. (2.3). This concludes the proof.

Remark. *Above discussion has established the case when g is nonnegative. For general case, we define* $E[g(X)] = E[g^+(X)] - E[g^-(X)]$.

References

[1] Eugene Wong and Bruce Hajek, *Stochastic processes in engineering systems*, Springer Science & Business Media, 1985.

[2] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.

[3] Allan Gut, *Probability: A graduate course*, 2nd Edition, Springer & Verlag, 2012.

[4] Bruce Hajek, *Random processes for engineers*, Cambridge University Press, 2015.