KTH, SCHOOL OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

Independence between Sigma-fields

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September 7, 2018

1 PROBLEM STATEMENT

Probability space is given as $(\Omega, \mathscr{F}, \mathbf{P})$. Suppose $X_1, X_2, ..., X_n$ are *mutually independent* random variables (r.v.'s). To distinguish between *mutual independence* and *pairwise independence*, refer to Definition 3.1 and its follow up Warning in [1]. Denote the sigma-field generated by X_i as $\sigma(X_i)$ and the sigma field generated by a sequence of of r.v.'s $(X_i)_{i=1}^m$ as $\sigma((X_i)_{i=1}^m)$. Prove that the sigma-filds $\sigma((X_i)_{i=1}^{n-1})$ and $\sigma(X_n)$ are independent.

2 ELABORATION

To show $\sigma((X_i)_{i=1}^{n-1})$ and $\sigma(X_n)$ are independent, by Definition 1.6.1, [2], we need to show that $\forall A \in \sigma((X_i)_{i=1}^{n-1})$ and $\forall B \in \sigma(X_n)$, the condition

$$\mathbf{P}(A \cap B) = \mathbf{P}(A) \cap \mathbf{P}(B) \tag{2.1}$$

always holds. We proceed by four steps.

1. For those $A^{(1)} \in \cup_{i=1}^{n-1}\sigma(X_i)$, by independence between X_i and X_n where $i = 1, 2, ..., n-1$, we know Eq. (2.1) holds $\forall B \in \sigma(X_n)$.

2. For those $A^{(2)} = \cup_{k=1}^{j} A_k^{(1)}$ where $A_k^{(1)} \in \cup_{i=1}^{n-1}\sigma(X_i)$ and $A^{(2)} \notin \cup_{i=1}^{n-1}\sigma(X_i)$, we first group $A_k^{(1)}$ by which $\sigma(X_i)$ it belongs to. Then $A^{(2)} = \cup_{m=1}^{n-1} S_m^{(1)}$ where $S_m^{(1)} = \cup_{p\in\mathscr{P}(m)} A_p^{(1)}$ where $\mathscr{P}(m) = \{p \in \mathbf{N} | A_p^{(1)} \in \sigma(X_m)\}$. Then as a direct result, we have $S_m^{(1)} \in \sigma(X_m)$. So our goal is to prove
$$\mathbf{P}(A^{(2)}B) = \mathbf{P}(A^{(2)})\mathbf{P}(B).$$

Since $A^{(2)} = \cup_{m=1}^{n-1} S_m^{(1)}$, we need to show

$$\mathbf{P}\big((\cup_{m=1}^{n-1} S_m^{(1)}) \cap B\big) = \mathbf{P}(\cup_{m=1}^{n-1} S_m^{(1)})\mathbf{P}(B). \tag{2.2}$$

Then we start with the case $\mathbf{P}\big((\cup_{m=1}^{2} S_m^{(1)}) \cap B\big) = \mathbf{P}(\cup_{m=1}^{2} S_m^{(1)})\mathbf{P}(B)$.

$$\begin{aligned}\mathbf{P}\big((S_1^{(1)} \cup S_2^{(1)}) \cap B\big) &= \mathbf{P}\big((S_1^{(1)} \cap B) \cup (S_2^{(1)} \cap B)\big)\\ &= \mathbf{P}(S_1^{(1)} \cap B) + \mathbf{P}(S_2^{(1)} \cap B) - \mathbf{P}\big((S_1^{(1)} \cap B) \cap (S_2^{(1)} \cap B)\big)\\ &= \mathbf{P}(S_1^{(1)} \cap B) + \mathbf{P}(S_2^{(1)} \cap B) - \mathbf{P}(S_1^{(1)} \cap S_2^{(1)} \cap B).\end{aligned} \tag{2.3}$$

Since $X_1, ..., X_n$ are mutually independent, Eq. (2.3) can proceed as follows.

$$\begin{aligned}\mathbf{P}\big((S_1^{(1)} \cup S_2^{(1)}) \cap B\big) &= \mathbf{P}(S_1^{(1)})\mathbf{P}(B) + \mathbf{P}(S_2^{(1)})\mathbf{P}(B) - \mathbf{P}(S_1^{(1)})\mathbf{P}(S_2^{(1)})\mathbf{P}(B)\\ &= \big(\mathbf{P}(S_1^{(1)}) + \mathbf{P}(S_2^{(1)}) - \mathbf{P}(S_1^{(1)})\mathbf{P}(S_2^{(1)})\big)\mathbf{P}(B)\\ &= \mathbf{P}(S_1^{(1)} \cup S_2^{(1)})\mathbf{P}(B).\end{aligned} \tag{2.4}$$

So we show Eq. (2.2) for the case $\cup_{m=3}^{n-1} S_m^{(1)} = \emptyset$. To proceed to the case $\mathbf{P}\big((\cup_{m=1}^{3} S_m^{(1)}) \cap B\big) = \mathbf{P}(\cup_{m=1}^{3} S_m^{(1)})\mathbf{P}(B)$, we need to show that $S_1^{(1)} \cup S_2^{(1)}$, $S_3^{(1)}$ and B are mutually independent. Indeed, we can show that

$$\begin{aligned}\mathbf{P}\big((S_1^{(1)} \cup S_2^{(1)}) \cap S_3^{(1)} \cap B\big) =&\mathbf{P}\big((S_1^{(1)} \cup S_2^{(1)}) \cap (S_3^{(1)} \cap B)\big)\\ =&\mathbf{P}\big((S_1^{(1)} \cap S_3^{(1)} \cap B) \cup (S_2^{(1)} \cap S_3^{(1)} \cap B)\big)\\ =&\mathbf{P}(S_1^{(1)} \cap S_3^{(1)} \cap B) + \mathbf{P}(S_2^{(1)} \cap S_3^{(1)} \cap B) -\\ &\mathbf{P}\big((S_1^{(1)} \cap S_3^{(1)} \cap B) \cap (S_2^{(1)} \cap S_3^{(1)} \cap B)\big)\\ =&\mathbf{P}(S_1^{(1)} \cap S_3^{(1)} \cap B) + \mathbf{P}(S_2^{(1)} \cap S_3^{(1)} \cap B) - \mathbf{P}(S_1^{(1)} \cap S_2^{(1)} \cap S_3^{(1)} \cap B).\end{aligned} \tag{2.5}$$

Since $X_1, ..., X_n$ are mutully independent, Eq. (2.5) can proceed as follows.

$$\begin{aligned}\mathbf{P}\big((S_1^{(1)} \cup S_2^{(1)}) \cap S_3^{(1)} \cap B\big) =&\big(\mathbf{P}(S_1^{(1)}) + \mathbf{P}(S_2^{(1)}) - \mathbf{P}(S_1^{(1)})\mathbf{P}(S_2^{(1)})\big)\mathbf{P}(S_3^{(1)})\mathbf{P}(B)\\ =&\big(\mathbf{P}(S_1^{(1)}) + \mathbf{P}(S_2^{(1)}) - \mathbf{P}(S_1^{(1)} \cap S_2^{(1)})\big)\mathbf{P}(S_3^{(1)})\mathbf{P}(B)\\ =&\mathbf{P}(S_1^{(1)} \cup S_2^{(1)})\mathbf{P}(S_3^{(1)})\mathbf{P}(B).\end{aligned} \tag{2.6}$$

Therefore, we can proceed like Eqs. (2.3) and (2.4) till $n-1$ to prove Eq. (2.2).

3. Denote the collection of $A^{(2)}$ as $\mathscr{C}$. Due to how $A^{(2)}$ is defined, we know $\mathscr{C} \cap \big(\cup_{i=1}^{n-1} \sigma(X_i)\big) = \emptyset$. Define $A^{(3)} = \cup_{j=1}^{\infty} A_j$, $A_j \in \mathscr{C} \cup \big(\cup_{i=1}^{n-1} \sigma(X_i)\big)$. Denote $\cup_{j=1}^{k} A_j = S_k$. Then we know $S_k \in \mathscr{C} \cup \big(\cup_{i=1}^{n-1} \sigma(X_i)\big)$ and $S_1 \subseteq S_2 \subseteq ... \subseteq S_n \subseteq ....$ Then we know $\cup_{j=1}^{k} A_j = \cup_{j=1}^{k} S_j$ and $\cup_{j=1}^{\infty} A_j = \cup_{j=1}^{\infty} S_j$.

$$\begin{aligned}\mathbf{P}(A^{(3)} \cap B) &= \mathbf{P}\big((\cup_{j=1}^{\infty} A_j) \cap B\big)\\ &= \mathbf{P}\big((\cup_{j=1}^{\infty} S_j) \cap B\big).\end{aligned} \tag{2.7}$$

We can show that $(\cup_{j=1}^{k} S_j) \cap B = \cup_{j=1}^{k} (S_j \cap B)$ and $(\cup_{j=1}^{\infty} S_j) \cap B = \cup_{j=1}^{\infty} (S_j \cap B)$. Denote $C_j = S_j \cap B$. Then we have $C_1 \subseteq C_2 \subseteq ... \subseteq C_n \subseteq ....$

$$\begin{aligned} \mathbf{P}(A^{(3)} \cap B) &= \mathbf{P}\big(\cup_{j=1}^{\infty} (S_j \cap B)\big) \\ &= \mathbf{P}(\cup_{j=1}^{\infty} C_j). \end{aligned} \tag{2.8}$$

Due to continuity from below in [2], as a direct result of $C_1 \subseteq C_2 \subseteq ... \subseteq C_n \subseteq ...$, we have

$$\begin{aligned} \mathbf{P}(A^{(3)} \cap B) &= \lim_{j\to\infty} \mathbf{P}(C_j) \\ &= \lim_{j\to\infty} \mathbf{P}(S_j \cap B). \end{aligned} \tag{2.9}$$

Since $S_j \in \mathscr{C} \cup \big(\cup_{i=1}^{n-1} \sigma(X_i)\big)$, by the proof of step 1 and 2, we have $\mathbf{P}(S_j \cap B) = \mathbf{P}(S_j)\mathbf{P}(B)$ $\forall j$. Therefore, we can proceed as follows.

$$\begin{aligned} \mathbf{P}(A^{(3)} \cap B) &= \lim_{j\to\infty} \big(\mathbf{P}(S_j)\mathbf{P}(B)\big) \\ &= \big(\lim_{j\to\infty} \mathbf{P}(S_j)\big)\mathbf{P}(B) \\ &= \mathbf{P}(\cup_{j=1}^{\infty} S_j)\mathbf{P}(B). \end{aligned} \tag{2.10}$$

Since $A^{(3)} = \cup_{j=1}^{\infty} S_j$, we have the result.

4. Similar steps can be applied to proceed for any $A \in \sigma((X_i)_{i=1}^{n-1})$ which does not belong to first three categories.

REFERENCES

[1] Jean Jacod and Philip Protter, *Probability essentials*, 2nd Edition, Springer Science & Business Media, 2012.

[2] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.