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Expectation and Independence

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1 Problem statement

Probability space is given as $(\Omega, \mathscr{F}, \mathbf{P})$. Suppose X and Y are random variables (r.v.'s) defined on the same probability space $(\Omega, \mathscr{F}, \mathbf{P})$. Suppose also that $E[X] < \infty$, $E[Y] < \infty$. By [Note 12] and [Note 13], we know $Z = XY$ is also a r.v. defined on the same probability space. Suppose $E[Z] < \infty$. If X and Y are independent, then by the Proposition B.21, P. 493 [1] or Exercise 3, P. 44 [2] we have

$$E[XY] = E[X]E[Y]. \tag{1.1}$$

Considering also the product probability space introduced in Section 9.1 and 9.2, P. 64, [3]. Elaborate the direct consequence from Eq. (1.1) regarding integration over the probability space.

2 Elaboration

Since $Z = XY$ is $\mathscr{F}$-measurable. Then we have

$$E[Z] = \int_\Omega Z d\mathbf{P} = \int_\Omega XY d\mathbf{P}. \tag{2.1}$$

Similarly, we have

$$E[X] = \int_\Omega X d\mathbf{P}, \tag{2.2}$$

$$E[Y] = \int_\Omega Y d\mathbf{P}. \tag{2.3}$$

Then due to Eq. (2.1), we have

$$\int_{\Omega} XY d\mathbf{P} = \int_{\Omega} X d\mathbf{P} \int_{\Omega} Y d\mathbf{P} = \int_{\Omega_X} \int_{\Omega_Y} XY d\mathbf{P}_X d\mathbf{P}_Y \tag{2.4}$$

where $\Omega_X = \Omega_Y = \Omega$ and $\mathbf{P}_X = \mathbf{P}_Y = \mathbf{P}$. Namely, if X and Y are independent, when calculating the expectation, even they are defined on the same probability space, they can be regarded as being defined on two probability spaces, both of which are duplicate of original probability space.

Remark. *This shall not come as a surprise. Althogh the symbol following the integration $\int$ is Ω, one shall be aware the integral is defined as Lebesgue integral, refer to [Hand Note 2], which is the limit of expectations of simple r.v.'s. For simple r.v.'s, the operation of taking expectation is conducted via events, so all that matters are the events A where $A \in \mathscr{F}$, not the elementary outcome ω, where $\omega \in \Omega$. As a result, if X and Y are independent, then $\forall A \in \mathscr{F}_X$ and $\forall B \in \mathscr{F}_Y$ where $\mathscr{F}_X$ and $\mathscr{F}_Y$ are sigma fields generated by X and Y respectively, we have*

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B).$$

So the relation expressed by Eq. (2.4) *would always hold for simple r.v.'s X_n and Y_n where $X_n \uparrow X$ and $Y_n \uparrow Y$. So would be their limits.*

References

[1] Tomas Björk, *Arbitrage theory in continuous time*, 3rd Edition, Oxford university press, 2009

[2] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.

[3] Allan Gut, *Probability: A graduate course*, 2nd Edition, Springer & Verlag, 2012.