KTH, School of Electrical Engineering and Computer Science

Regarding Conditional Expectation

Yuchao Li

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1 Problem statement

Probability space is given as $(\Omega, \mathscr{F}, \mathbf{P})$. Suppose X is a discrete random variable (r.v.) and $H(X)$ is a r.v.. For a fixed x, $H(x)$ may or may not be independent of X. Prove the following result

$$E[H(X)|X=x] = E[H(x)|X=x] \tag{1.1}$$

provided that $\mathbf{P}(X=x) > 0$.

2 Elaboration

First of all, we know $\{X=x\} \in \mathscr{F}$. By the definition of conditioning w.r.t. an event [1], for the left-hand side of Eq. (1.1) we have

$$\begin{aligned} E[H(X)|X=x] &= \frac{1}{\mathbf{P}(\{X=x\})}\int_{\{X=x\}} H(X)\,d\mathbf{P} \\ &= \frac{1}{\mathbf{P}(\{X=x\})}\int_{\Omega} \chi_{\{X=x\}} H(X)\,d\mathbf{P}. \end{aligned} \tag{2.1}$$

where $\chi_{\{X=x\}}(\omega)$ is the indicator function. Similarly, for the right-hand side of Eq. (1.1), we have

$$\begin{aligned} E[H(x)|X=x] &= \frac{1}{\mathbf{P}(\{X=x\})}\int_{\{X=x\}} H(x)\,d\mathbf{P} \\ &= \frac{1}{\mathbf{P}(\{X=x\})}\int_{\Omega} \chi_{\{X=x\}} H(x)\,d\mathbf{P}. \end{aligned} \tag{2.2}$$

Since we have

$$\chi_{\{X=x\}}(\omega)H(X)(\omega) = \chi_{\{X=x\}}(\omega)H(x)(\omega),\ \forall \omega \in \Omega,$$

then the integrals in Eqs. (2.1) and (2.2) are equal.

REFERENCES

[1] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.