KTH, School of Electrical Engineering and Computer Science

Random Variables Generated by Indexing

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1 Problem statement

Probability space is given as $(\Omega, \mathscr{F}, \mathbf{P})$. Suppose I, X_1, $X_2, ..., X_n, ...$ are independent positive integar valued random variables (r.v.). Prove the following are also r.v.'s.

1. X_I;
2. $S_n = \sum_{i=1}^{n} X_i$ for any fixed n;
3. S_I.

2 Elaboration

To show X_I, S_n and S_I are r.v.'s, we need to show they are $\mathscr{F}$-measurable.

1. Since I, X_1, $X_2, ..., X_n, ...$ are positive integar valued, so we have $X_I : \Omega \to \mathbf{N}$ where $\mathbf{N}$ is the set of positive integers. Then we have
$$\{X_I = x\} = \cup_{i=1}^{\infty}(\{X_i = x\} \cap \{I = i\}). \tag{2.1}$$
Since X_i is r.v. $\forall i \in \mathbf{N}$, $\{X_i = x\} \in \mathscr{F}$. Similarly, $\{I = i\} \in \mathscr{F}$. Since $\mathscr{F}$ is a sigma field, by closure under intersection, we know $\{X_i = x\} \cap \{I = i\} \in \mathscr{F}$. By closure under countable union, $\{X_I = x\} \in \mathscr{F}$. Therefore, X_I is a r.v..

2. Here we provide a simple elaboration to give a flavor of the proof. Consider the case where $n = 2$ and we need to check, for example, if $\{S_2 = 3\} \in \mathscr{F}$. Since $3 = 1 + 2 = 2 + 1$, therefore, we have
$$\{S_2 = 3\} = (\{X_1 = 1\} \cap \{X_2 = 2\}) \cup (\{X_1 = 2\} \cap \{X_2 = 1\}). \tag{2.2}$$

As a result, we have $\{S_2 = 3\} \in \mathscr{F}$. Generally for $\{S_n = i\}$, we can similarly distribute i to $X_1, X_2, ..., X_n$ and then it can be shown that $\{S_n = i\} \in \mathscr{F}$

3. By part 2, we know S_n is a positive integered r.v. for any fixed n. Then use the argument in part 1, where X_i is replaced by S_i, we get the proof.

REFERENCES

[1] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.