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Monotonicity of Lebesgue Integration

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1 Problem statement

Denote a probability space as $(\Omega, \mathscr{F}, \mathbf{P})$ and a r.v. defined upon it as X which is $\mathscr{G}$-measurable where $\mathscr{G} \subseteq \mathscr{F}$. In the proof of Lemma 3.5.1, P.96 in [1], it states that

$$\varepsilon \mathbf{P}(X \geq \varepsilon) = \int_{\{X \geq \varepsilon\}} \varepsilon d\mathbf{P} \leq \int_{\{X \geq \varepsilon\}} X d\mathbf{P}. \tag{1.1}$$

Suppose X is nonnegative random variable and $\varepsilon \geq 0$. Prove above result by its nature, namely considering the fact that the integral above is Lebesgue integration.
In addition, if $\mathbf{P}(X > \varepsilon) > 0$, then we have

$$\varepsilon \mathbf{P}(X \geq \varepsilon) = \int_{\{X \geq \varepsilon\}} \varepsilon d\mathbf{P} < \int_{\{X \geq \varepsilon\}} X d\mathbf{P}. \tag{1.2}$$

2 Elaboration

The first two steps in the following prove Eq. (1.1). The third step proves Eq. (1.2).

1. First, we prove the case where X is bounded.

   Denote $E = [\varepsilon, \infty) \in \mathscr{B}$. Then, since X is a random variable, we have $A \in \mathscr{G}$ where event A is given as $A = X^{-1}(E)$. Then the first integral in Eq. (1.1) is a Lebesgue integral of a simple random variable, and the random variable is defined via indicator function as $\varepsilon\chi_A$. Then we have

$$\int_{\{X \geq \varepsilon\}} X d\mathbf{P} = E[\chi_A X] \tag{2.1}$$

As for the second integral, refer to [Hand Note 2] and [1, 2, 3] for its definition. Note that as a Lesbegue integral, it is defined as a limit of the 'expectation' (over certain event, in this case A) of a series of simple random variables, given as

$$X_n(\omega) = \begin{cases} \frac{k-1}{2^n}, & \frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n}, \, k = 1, 2, ..., n2^n \\ n, & X(\omega) \geq n. \end{cases} \tag{2.2}$$

Denote $E_{n,k} = [\frac{k-1}{2^n}, \frac{k}{2^n}) \cap E$. Due to that X is bounded, we suppose it is bounded by $M \in \mathbf{N}_+$. Then for $n \geq M$, we define Z_n and Y_n as

$$Z_n(\omega) = \begin{cases} \frac{1}{2^n}, & X(\omega) < n \\ 0, & X(\omega) \geq n. \end{cases} \tag{2.3}$$

$$Y_n(\omega) = \begin{cases} \frac{k}{2^n}, & \frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n}, \, k = 1, 2, ..., n2^n \\ n, & X(\omega) \geq n, \end{cases} \tag{2.4}$$

Then we have $Y_n = X_n + Z_n$ and $X_n \geq 0$, $Z_n \geq 0$. Then for $n \geq M$, due to $\mathbf{P}(\{X(\omega) \geq n\}) = 0$, we have

$$\begin{aligned} E[\chi_A X_n] &= \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{P}(\chi_A(\omega) X(\omega) \in E_{n,k}) + n\mathbf{P}(\{X(\omega) \geq n\}) \qquad (2.5a) \\ &= \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{P}(\chi_A(\omega) X(\omega) \in E_{n,k}) \end{aligned}$$

$$\begin{aligned} E[\chi_A Z_n] &= \sum_{k=1}^{n2^n} \frac{1}{2^n} \mathbf{P}(\chi_A(\omega) X(\omega) \in E_{n,k}) + n\mathbf{P}(\{X(\omega) \geq n\}) \qquad (2.5b) \\ &= \sum_{k=1}^{n2^n} \frac{1}{2^n} \mathbf{P}(\chi_A(\omega) X(\omega) \in E_{n,k}) \end{aligned}$$

Due to linearity in Theorem 4.1, P.49, [2], since $\chi_A Y_n = \chi_A X_n + \chi_A Z_n$, we have

$$E[\chi_A Y_n] = E[\chi_A X_n] + E[\chi_A Z_n]. \tag{2.6}$$

Take limits on both sides of Eq. (2.6), due to definition of Lebesgue integral and the fact that $\mathbf{P}(\{X(\omega) \geq n\}) = 0$, we have

$$\lim_{n \to \infty} E[\chi_A X_n] = \int_{\{X \geq \varepsilon\}} X d\mathbf{P}. \tag{2.7}$$

Since X is bounded by M, then $Z_n^{-1}([0, M]) = \Omega$, then

$$\lim_{n \to \infty} E[\chi_A Z_n] = \lim_{n \to \infty} \frac{1}{2^n} = 0. \tag{2.8}$$

As a result, we have

$$\lim_{n \to \infty} E[\chi_A Y_n] = \lim_{n \to \infty} E[\chi_A X_n] + \lim_{n \to \infty} E[\chi_A Z_n] = \lim_{n \to \infty} E[\chi_A X_n] = \int_{\{X \geq \varepsilon\}} X d\mathbf{P}. \tag{2.9}$$

It can be seen that $Y_n > X_n \geq \varepsilon$, then we have

$$\sum_{k=1}^{n2^n} \frac{k}{2^n} \mathbf{P}(\chi_A(\omega)X(\omega) \in E_{n,k}) + n\mathbf{P}(\{X(\omega) \geq n\}) > \sum_{k=1}^{n2^n} \varepsilon \mathbf{P}(\chi_A(\omega)X(\omega) \in E_{n,k}) + \varepsilon \mathbf{P}(\{X(\omega) \geq n\}) \tag{2.10}$$

for $\forall n \geq M$. Then we have

$$\sum_{k=1}^{n2^n} \frac{k}{2^n} \mathbf{P}(\chi_A(\omega)X(\omega) \in E_{n,k}) > \sum_{k=1}^{n2^n} \varepsilon \mathbf{P}(\chi_A(\omega)X(\omega) \in E_{n,k}) = \int_{\{X \geq \varepsilon\}} \varepsilon d\mathbf{P} \tag{2.11}$$

Take limit of $n \to \infty$ on both sides of Eq. (2.11), we have the result.

2. Then we prove the case where $X \in \mathbf{R}$, which is more general than previous result.

For $n \geq \varepsilon$, define W_n as

$$W_n(\omega) = \begin{cases} 0, & X(\omega) < \varepsilon \text{ or } X(\omega) \geq n + \varepsilon, \\ \frac{k-1}{2^n} + \varepsilon, & \frac{k-1}{2^n} + \varepsilon \leq X(\omega) < \frac{k}{2^n} + \varepsilon, \ k = 1, 2, ..., n2^n. \end{cases} \tag{2.12}$$

Then for any $W_n(\omega)$, denote $F_{n,k} = [\frac{k-1}{2^n} + \varepsilon, \frac{k}{2^n} + \varepsilon)$ for $k = 1, 2, ..., n2^n$ and $F_{n,n2^n+1} = [n + \varepsilon, \infty)$

$$E[W_n] + \varepsilon \mathbf{P}(\{X(\omega) \geq n + \varepsilon\}) = \sum_{k=1}^{n2^n} \big((\frac{k-1}{2^n} + \varepsilon) \mathbf{P}(X(\omega) \in F_{n,k}) \big) + \varepsilon \mathbf{P}(\{X(\omega) \geq n + \varepsilon\}) \tag{2.13}$$

Since $\frac{k-1}{2^n} + \varepsilon \geq \varepsilon$ for $k = 1, 2, ..., n2^n$, then we have

$$E[W_n] + \varepsilon \mathbf{P}(\{X(\omega) \geq n + \varepsilon\}) \geq \varepsilon \big(\sum_{k=1}^{n2^n} \mathbf{P}(X(\omega) \in F_{n,k}) + \mathbf{P}(\{X(\omega) \geq n + \varepsilon\}) \big) = \varepsilon \mathbf{P}(X \geq \varepsilon). \tag{2.14}$$

Take limits on both sides of Eq. (2.14). Since W_n converge to $\chi_A X$ from below, due to Theorem 4.2 (Consistency), P.50, [2], we have

$$\lim_{n \to \infty} E[W_n] = E[\chi_A X] = \int_{\{X \geq \varepsilon\}} X d\mathbf{P}. \tag{2.15}$$

As for the limit of $\mathbf{P}(\{X(\omega) \geq n + \varepsilon\})$, we have

$$\lim_{n \to \infty} \mathbf{P}(\{X(\omega) \geq n + \varepsilon\}) = \mathbf{P}(\cap_{i=1}^{\infty} \{X(\omega) \geq i + \varepsilon\}) = \mathbf{P}(\emptyset) = 0 \tag{2.16}$$

where the first equality is due to Theorem 1.4.9 (continuity from above) [1], while the second is due to that $X \in \mathbf{R}$. As a result, we have at last

$$\int_{\{X \geq \varepsilon\}} X d\mathbf{P} \geq \varepsilon \mathbf{P}(X \geq \varepsilon). \tag{2.17}$$

3. If $\mathbf{P}(X > \varepsilon) > 0$, denote $B = X^{-1}((\varepsilon, \infty))$ and $C = X^{-1}(\varepsilon)$. Then we know $B \cup C = A$ and $\mathbf{P}(B) > 0$. Since the following formula holds

$$\chi_A(\omega) = \chi_B(\omega) + \chi_C(\omega),\ \forall \omega \in \Omega, \tag{2.18}$$

then we have

$$\int_{\{X \geq \varepsilon\}} X d\mathbf{P} = E[\chi_A X] = E[(\chi_B + \chi_C)X] = E[\chi_B X] + E[\chi_C X] \tag{2.19}$$

Since $E[\chi_B X] = \int_{\{X>\varepsilon\}} X d\mathbf{P}$ and $E[\chi_C X] = \int_{\{X=\varepsilon\}} X d\mathbf{P}$ by definition, according to Section 1.12.3, P. 44 [1], then we have

$$\int_{\{X \geq \varepsilon\}} X d\mathbf{P} = \int_{\{X>\varepsilon\}} X d\mathbf{P} + \int_{\{X=\varepsilon\}} X d\mathbf{P} \tag{2.20}$$

Apparently, we have

$$\varepsilon \mathbf{P}(X = \varepsilon) = \int_{\{X=\varepsilon\}} X d\mathbf{P}. \tag{2.21}$$

Therefore the part left for proof is to show that

$$\varepsilon \mathbf{P}(X > \varepsilon) < \int_{\{X>\varepsilon\}} X d\mathbf{P}. \tag{2.22}$$

Define $T_n(\omega)$ as

$$T_n(\omega) = \begin{cases} 0, & \chi_B(\omega)X(\omega) < \varepsilon, \\ \frac{k-1}{2^n} + \varepsilon, & \frac{k-1}{2^n} + \varepsilon \leq \chi_B(\omega)X(\omega) < \frac{k}{2^n} + \varepsilon,\ k = 1, 2, ..., n2^n, \\ n + \varepsilon, & \chi_B(\omega)X(\omega) \geq n + \varepsilon. \end{cases} \tag{2.23}$$

Then we have

$$E[T_n] = \sum_{k=1}^{n2^n+1} \Big((\frac{k-1}{2^n} + \varepsilon) \mathbf{P}(\chi_B(\omega)X(\omega) \in F_{n,k}) \Big) \tag{2.24}$$

Since $\cap_{n=1}^{\infty}(F_{n,1} \cap (\varepsilon, \infty)) = \emptyset$, then we have

$$\mathbf{P}\Big(\chi_B X \in \cap_{n=1}^{\infty}\big(F_{n,1} \cap (\varepsilon, \infty)\big)\Big) = 0. \tag{2.25}$$

Since $F_{n,1} \cap (\varepsilon, \infty)$ is decreasing with the increase of n, we have

$$\lim_{n\to\infty} \mathbf{P}\Big(\chi_B X \in \big(F_{n,1} \cap (\varepsilon, \infty)\big)\Big) = \mathbf{P}\Big(\chi_B X \in \cap_{n=1}^{\infty}\big(F_{n,1} \cap (\varepsilon, \infty)\big)\Big) = 0. \tag{2.26}$$

Then we know $\forall \xi < \mathbf{P}(X > \varepsilon)$, $\exists N$ such that $\mathbf{P}\Big(\chi_B X \in \big(F_{n,1} \cap (\varepsilon, \infty)\big)\Big) < \xi$ holds $\forall n \geq N$. Then as a result, we have

$$\mathbf{P}\Big(\chi_B X \in \big((\varepsilon, \infty) \setminus F_{n,1}\big)\Big) = \mathbf{P}(B) - \mathbf{P}\Big(\chi_B X \in \big(F_{n,1} \cap (\varepsilon, \infty)\big)\Big) > \mathbf{P}(B) - \xi > 0 \tag{2.27}$$

holds $\forall n \geq N$. On the other hand, we have

$$\begin{aligned}
\lim_{n\to\infty} E[T_n] &\geq \sum_{k=1}^{N2^N+1} \big((\frac{k-1}{2^N}+\varepsilon)\mathbf{P}(\chi_B(\omega)X(\omega)\in F_{N,k})\big) \\
&\geq \sum_{k=2}^{N2^N+1} \big((\frac{1}{2^N}+\varepsilon)\mathbf{P}(\chi_B(\omega)X(\omega)\in F_{N,k})\big)+\varepsilon\mathbf{P}(\chi_B(\omega)X(\omega)\in F_{N,1})\big) \\
&= (\frac{1}{2^N}+\varepsilon)\sum_{k=2}^{N2^N+1} \mathbf{P}(\chi_B(\omega)X(\omega)\in F_{N,k})+\varepsilon\mathbf{P}(\chi_B(\omega)X(\omega)\in F_{N,1})\big) \\
&= (\frac{1}{2^N}+\varepsilon)\mathbf{P}\Big(\chi_B(\omega)X(\omega)\in\big((\varepsilon,\infty)\setminus F_{N,1}\big)\Big)+\varepsilon\mathbf{P}(\chi_B(\omega)X(\omega)\in F_{N,1})\big) \\
&> \varepsilon\mathbf{P}\big(\chi_B(\omega)X(\omega)\in(\varepsilon,\infty)\big) \qquad (2.28)
\end{aligned}$$

where the first inequality is due to that T_n as simple r.v.'s are increasing and as a result of [Hand Note 1] $E[T_n]$ is increasing; the last inequality is due to $\frac{1}{2^N}+\varepsilon>\varepsilon$ and $\mathbf{P}\Big(\chi_B(\omega)X(\omega)\in\big((\varepsilon,\infty)\setminus F_{N,1}\big)\Big)>0$. Therefore, we estabilished the result that

$$\varepsilon\mathbf{P}(X>\varepsilon)<\lim_{n\to\infty} E[T_n]. \qquad (2.29)$$

Due to Theorem 4.2 (Consistency), P.50, [2], $\lim_{n\to\infty} E[T_n]=E[\chi_B X]$. Therefore, the proof is done.

References

[1] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.

[2] Allan Gut, *Probability: A graduate course*, 2nd Edition, Springer & Verlag, 2012.

[3] Bruce Hajek, *Random processes for engineers*, Cambridge University Press, 2015.