KTH, SCHOOL OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

Discrete Part of a Measure

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1 PROBLEM STATEMENT

We denote a probability measure on $(\mathbf{R}, \mathscr{B})$ as μ and its probability mass function as p where

$$p(x) \overset{\text{def}}{=} \mu(\{x\}), \; -\infty < x < \infty. \tag{1.1}$$

Due to Lemma 2.5.2 in [1], we can define a discrete part of μ as

$$\mu_D(A) = \sum_{x \in A | p(x) > 0} p(x), \; A \in \mathscr{B}. \tag{1.2}$$

Prove that μ_D is a measure on $(\mathbf{R}, \mathscr{B})$.

2 ELABORATION

Apparently, the part we need to prove is

$$\mu_D(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu_D(A_i) \tag{2.1}$$

where $(A_i)_{i=1}^{\infty}$ is pairwise disjoint and $A_i \in \mathscr{B} \; \forall i$. To do so, we denote $A = \cup_{i=1}^{\infty} A_i$ and $B = \{x \in A : p(x) > 0\}$. Then $B = \{x \in A : \mu(\{x\}) > 0\}$. Apparently, it can be shown that $B = \{x \in A : \mu(\{x\}) > 0\} \in \mathscr{B} \; \forall A \in \mathscr{B}$ (by definition). Due to Lemma 2.5.2, B can only have countably many elements. If the cardinality of B is infinite, we order the elements of B in certain way as $x_1, x_2, ..., x_n, ...$, namely

$$B = \{x_1, x_2, ..., x_n, ...\} = \cup_{i=1}^{\infty} \{x_i\} \tag{2.2}$$

Then we have

$$\mu_D(A) = \sum_{x\in A|p(x)>0} p(x) = \sum_{x\in A|\mu(\{x\})>0} \mu(\{x\}) = \sum_{i=1}^{\infty} \mu(\{x_i\}) = \mu(\cup_{i=1}^{\infty}\{x_i\}) = \mu(B) \tag{2.3}$$

where the first equality is due to definition of μ_D (1.2); the second equality is due to definition of p (1.1); the third equality is due to the fact that commutation of series does not change its sum, refer to the footnote in P.22 of [2]; the fourth equality is due to that μ is a measure and $(\{x_i\})_{i=1}^{\infty}$ are pairwise disjoint. If the cardinality of A is finite, then the proof is trivial.
To conclude, we have

$$\mu_D(A) = \mu(B) \tag{2.4}$$

holds $\forall A \in \mathscr{B}$ and $B = \{x \in A : p(x) > 0\}$ shall belong to $\mathscr{B}$ and therefore $\mu(B)$ is defined. Denote $B_i = \{x \in A_i : p(x) > 0\}$. Since $A = \cup_{i=1}^{\infty} A_i$, we can prove $B = \cup_{i=1}^{\infty} B_i$ (by definition) and B_i as pairwise disjoint. As a result of μ as a measure, we have

$$\mu(B) = \sum_{i=1}^{\infty} \mu(B_i). \tag{2.5}$$

Since $\mu_D(A) = \mu(B)$ and $\mu_D(A_i) = \mu(B_i)$ $\forall i$, we prove Eq . (2.1).

REFERENCES

[1] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.

[2] Patrick Billingsley, *Probability and measure*, 3rd Edition, John Wiley & Sons, 1995.