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Expectation of Random Variables with respect to Dsitribution

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1 Problem statement

Probability space is given as $(\Omega, \mathscr{F}, \mathbf{P})$ with elementary outcome ω. For a nonnegative random variable X, its expectation can be calculated by

$$E[X] = \int_0^\infty x dF_X(x). \tag{1.1}$$

2 Elaboration

1. First, we recall the definition of the expection of a nonnegative random variable, namely $E[X]$, given X. This follows the approach of [1].

   For the given random variable X, we define X_n as

$$X_n(\omega) = \begin{cases} \frac{k-1}{2^n}, & \frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n}, k = 1, 2, ..., n2^n \\ n, & X(\omega) \geq n. \end{cases} \tag{2.1}$$

   Denote $f_n : \mathbf{R}_+ \cup \{0\} \to \mathbf{R}$ as

$$f_n(x) = \begin{cases} \frac{k-1}{2^n}, & \frac{k-1}{2^n} \leq x < \frac{k}{2^n}, k = 1, 2, ..., n2^n, \\ n, & x \geq n. \end{cases} \tag{2.2}$$

   we can show that f_n is a Borel function. Since X is a random variable, then $X_n = f_n(X)$ is a random variable. Further we can see that in fact X_n is a simple random variable.

Then, it can be shown (refer to [1, 2]) that $X_n(\omega) \to X(\omega)$ as $n \to \infty\ \forall \omega$. Based on X_n, the expectation $E[X]$ is defined as a Lebesgue integral (refer to [1]) that

$$E[X] = \lim_{n\to\infty} \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{P}(\frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n}). \tag{2.3}$$

Note that compared to [2], the term $n\mathbf{P}(X \geq n)$ does not show up in the definition. Clearly, the term makes no difference if $X(\omega) < M\ \forall \omega$. In this case, we would have that $\lim_{n\to\infty} n\mathbf{P}(X \geq n) = 0$. Since $\exists N > M$ such that $\forall n > N$ that $X < n\ \forall \omega$. Therefore, $\mathbf{P}(X \geq n) = 0\ \forall n > N$. As a result, $\lim_{n\to\infty} n\mathbf{P}(X \geq n) = 0$. If $X(\omega) < M\ \forall \omega$ does not hold, due to Theorem 4.2 (Consistency), P.50, [1], we have

$$\lim_{n\to\infty} E[X_n] = E[X] \tag{2.4}$$

where on the left hand side $n\mathbf{P}(X \geq n)$ appears while on the right-hand side, due to definition Eq. (2.3), $n\mathbf{P}(X \geq n)$ does not appear.

Regarding the convergence related to the definition, refer to [1, 2, 3].

Eq. (2.3) by definition is also rewritten as

$$E[X] = \int_\Omega X(\omega) d\mathbf{P}(\omega) \ \text{(Lebesgue)}. \tag{2.5}$$

2. Second, we show that for two random variables, X defined on $(\Omega, \mathscr{F}, \mathbf{P})$ and Y defined on $(\Omega', \mathscr{F}', \mathbf{P}')$. If $F_X(a) = F_Y(a)\ \forall a \in \mathbf{R}$, then $E[X] = E[Y]$.

   Since $F_X(a) = F_Y(a)$, namely $\mathbf{P}(X \leq a) = \mathbf{P}'(Y \leq a)$. One can show for any Borel set A, $\mathbf{P}(A) = \mathbf{P}'(A)$. As a result, $\mathbf{P}(\frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n}) = \mathbf{P}'(\frac{k-1}{2^n} \leq Y(\omega') < \frac{k}{2^n})\ \forall n, k$. Therefore, the constructed X_n and Y_n used to define $E[X]$ and $E[Y]$, as shown in the previous step, would have the property that

$$E[X_n] = E[Y_n],\ \forall n. \tag{2.6}$$

   Take limits on both sides of Eq. (2.6), we get $E[X] = E[Y]$, namely

$$\text{(Lebesgue)} \int_\Omega X(\omega) d\mathbf{P}(\omega) = \int_{\Omega'} Y(\omega') d\mathbf{P}'(\omega') \ \text{(Lebesgue)} \tag{2.7}$$

3. Then we define the canoical random variable. Since X is a random variable on $(\Omega, \mathscr{F}, \mathbf{P})$ with distribution function F_X. Then we can find a probability space $(\mathbf{R}, \mathscr{B}, \tilde{\mathbf{P}})$ where $\mathscr{B}$ is Borel sigma field and $\tilde{\mathbf{P}}((a, b]) = F_X(b) - F_X(a)$. By extension theorem, $\tilde{\mathbf{P}}$ is well defined. Then define $Y : \mathbf{R} \to \mathbf{R}$ as a random variable on $(\mathbf{R}, \mathscr{B}, \tilde{\mathbf{P}})$ and is given by $Y(y) = y$. Then the distribution function of Y, written as F_Y, would have the property that $F_Y(a) = F_X(a)\ \forall a \in \mathbf{R}$. As a direct result of step 2, we have

$$\text{(Lebesgue)} \int_{-\infty}^{\infty} Y(r) d\tilde{\mathbf{P}}(r) = \int_\Omega X(\omega) d\mathbf{P}(\omega) \ \text{(Lebesgue)}. \tag{2.8}$$

   Please refer to Prop. 1.4 [3] for origin.

4. Finally, according to Sec. 11.5.4 [3], if F_X is a distribution function, a Lebesgue-Stieltjes integration of F_X is defined by the Lebesgue integration of its caniocal random variable, namely

$$\text{(Lebesgue-Stieltjes)} \int_{-\infty}^{\infty} g(x) dF_X(x) = \int_{-\infty}^{\infty} g\big(Y(r)\big) d\tilde{\mathbf{P}}(r) \text{ (Lebesgue).} \tag{2.9}$$

where $Y(r) = r$ as introduced above. Set $g(x) = x$, then we get

$$\text{(Lebesgue-Stieltjes)} \int_{-\infty}^{\infty} x dF_X(x) = \int_{-\infty}^{\infty} Y(r) d\tilde{\mathbf{P}}(r) \text{ (Lebesgue).} \tag{2.10}$$

Combine Eqs. (2.5), (2.8) and (2.10), we have

$$E[X] = \int_{-\infty}^{\infty} x dF_X(x) \text{ (Lebesgue-Stieltjes).} \tag{2.11}$$

Note in the Eq. (2.9), if g is continuous (in our case, $g(x) = x$ is continuous) and the integral is finite, then the Lebesgue-Stieltjes integration and Riemann-Stieltjes integration (conventional one) agrees [3]. This is the reason we can calculate in most cases the expection $E[X]$ as

$$E[X] = \int_{-\infty}^{\infty} x dF_X(x) \text{ (Riemann-Stieltjes).} \tag{2.12}$$

Remark. *Above discussion has established the case when X is nonnegative. For general case, we define $E[X] = E[X^+] - E[X^-]$. Notice that the X_n used to setablish $E[X]$ in [3] is given as*

$$X_n(\omega) \begin{cases} \frac{k-1}{2^n}, & \frac{k-1}{2^n} \le X(\omega) < \frac{k}{2^n}, k = 1, 2, ..., n2^n \\ 0, & X(\omega) \ge n, \end{cases} \tag{2.13}$$

which is different from Eq. (2.1) *from [1]. And [3] gives a better definition of X^+ and X^-. However, it should be understood that double counting $\mathbf{P}(X = 0)$ in both $E[X^+]$ and $E[X^-]$ does not alter the true value of $E[X]$ since its contribution to $E[X]$ would be* 0 *anyway.*

REFERENCES

[1] Allan Gut, *Probability: A graduate course*, 2nd Edition, Springer & Verlag, 2012.

[2] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.

[3] Bruce Hajek, *Random processes for engineers*, Cambridge University Press, 2015.