KTH, School of Electrical Engineering and Computer Science

Karhunen-Loéve Expansion and Mean Square Integral

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1 Problem statement

Given a Hilbert space $L_2(\Omega, \mathscr{F}, \mathbf{P})$, define a sequence of continuous random processes (r.p.'s) $\{X_i(t)\}_{i=1}^{\infty}$ for $t \in [0, T]$. Besides, we assume that $\forall t$, it holds that $X_n(t) \xrightarrow{2} X(t)$. We further assume that $Y_n = \int_0^T X_n(t)dt$ exists $\forall n$ and $Y = \int_0^T X(t)dt$ exists where those integrals are mean square integrals, namely, for example, Y is defined as

$$\sum_{i=1}^{m} X(t_i)(t_i - t_{i-1}) \xrightarrow{2} \int_0^T X(t)dt$$

where $0 = t_0 < t_1 < ... < t_m = T$ and $\max_i |t_i - t_{i-1}| \to 0$ as $m \to \infty$, as in Definition 9.2.1, P. 237, [1]. Prove that if $X_n(t)$ is the partial sum of X's Karhunen-Loéve expansion, then it holds that $Y_n \xrightarrow{2} Y$. By stating that $X_n(t)$ is the partial sum of X's Karhunen-Loéve expansion, we mean that $X(t)$ is constructed in the following way. Given an i.i.d sequence of random variables (r.v.'s) $\{Z_i\}_{i=1}^{\infty}$ in $L_2(\Omega, \mathscr{F}, \mathbf{P})$ with zero means, $X_n(t)$ is defined as $X_n(t) = \sum_{i=1}^{n} \sqrt{\lambda_i} Z_i e_i(t)$ where $e_i : \mathbb{R} \to \mathbb{R}$ is some continuous function. If it holds that $\sum_{i=1}^{\infty} \lambda_i < \infty$, then $\{X_i(t)\}_{i=1}^{\infty}$ is convergent in $L_2(\Omega, \mathscr{F}, \mathbf{P})$ $\forall t$. The r.v. it converges to at time t is defined as $X(t)$. We also use the symbol $\sum_{i=1}^{\infty} \sqrt{\lambda_i} Z_i e_i(t)$ to denote $X(t)$. By such construction, all the assumptions above are fulfilled. For details of such construction, refer to Section 7.4.2, P. 196 and Example 9.1.8, P. 234, [1].

2 ELABORATION

We first compute $E[|Y_n - Y|^2]$ as follows.

$$\begin{aligned}
E[|Y_n - Y|^2] &= E[(\int_0^T X_n(t)dt - \int_0^T X(t)dt)^2] \\
&= E[(\int_0^T X_n(t) - X(t)dt)^2] \\
&= E[\big(\lim_{\triangle} \sum_{i=1}^{m} \big(X_n(t_i) - X(t_i)\big)(t_i - t_{i-1})\big)^2]
\end{aligned}$$

where in the second equality, we apply the linearity property of mean square integral, Theorem 9.2.2 (a), P. 239, [1], and the auxiliary notion $\lim_{\triangle}$ refers to mean square convergence as $0 = t_0 < t_1 < ... < t_m = T$ and $\max_i |t_i - t_{i-1}| \to 0$ as $m \to \infty$. Then we introduce some predefined partitions on $[0, T]$ fulfilling above requirements, denoted as $\{P_i\}_{i=1}^{\infty}$, and its corresponding r.v. sequences, $\{\Delta Y_{n,i}\}_{i=1}^{\infty}$, namely $\Delta Y_{n,m} = \sum_{i=1}^{m} \big(X_n(t_i) - X(t_i)\big)(t_i - t_{i-1})$. By definition, we have $\Delta Y_{n,m} \xrightarrow{2} \int_0^T X_n(t) - X(t)dt = Y_n - Y$ as $m \to \infty$. As a result, we have

$$E[|Y_n - Y|^2] = E[(\lim_{m\to\infty} \Delta Y_{n,m})^2].$$

By Theorem 7.3.1 (c) and Theorem 7.3.3, P. 195, [1] we know that if $X_k \xrightarrow{2} X$, then we have $E[X^2] = \lim_{\min(m,k)\to\infty} E[X_m X_k]$, then we proceed the calculation as

$$\begin{aligned}
E[|Y_n - Y|^2] &= \lim_{\min(m,k)\to\infty} E[\Delta Y_{n,m} \Delta Y_{n,k}] \\
&= \lim_{\min(m,k)\to\infty} E[\sum_{i=1}^{m}\sum_{j=1}^{k} \big(X_n(t_i) - X(t_i)\big)\big(X_n(t_j) - X(t_j)\big)(t_i - t_{i-1})(t_j - t_{j-1})] \\
&= \lim_{\min(m,k)\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{k} E[\big(X_n(t_i) - X(t_i)\big)\big(X_n(t_j) - X(t_j)\big)](t_i - t_{i-1})(t_j - t_{j-1}) \\
&= \lim_{\min(m,k)\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{k} \big(R_{X_n}(t_i, t_j) + R_X(t_i, t_j) \\
&\qquad - E[X_n(t_i)X(t_j)] - E[X(t_i)X_n(t_j)]\big)(t_i - t_{i-1})(t_j - t_{j-1}).
\end{aligned}$$

where $R_{X_n}(\cdot,\cdot)$ and $R_X(\cdot,\cdot)$ are autocorrelation functions of X_n and X respectively. Since $X_n(t) \xrightarrow{2} X(t)$, then Theorem 7.3.1 (d), P. 195, [1], we have

$$\begin{aligned}
E[X(t_i)X_n(t_j)] &= \lim_{m\to\infty} E[X_m(t_i)X_n(t_j)] \\
&= \lim_{l\to\infty} E[X_{n+l}(t_i)X_n(t_j)] \\
&= \lim_{l\to\infty} E[X_n(t_i)X_n(t_j) + \big(X_{n+l}(t_i) - X_n(t_i)\big)X_n(t_j)] \\
&= E[X_n(t_i)X_n(t_j)] + \lim_{l\to\infty} E[\big(X_{n+l}(t_i) - X_n(t_i)\big)X_n(t_j)] \\
&= R_{X_n}(t_i, t_j) + \lim_{l\to\infty} E[\big(X_{n+l}(t_i) - X_n(t_i)\big)X_n(t_j)].
\end{aligned}$$

Due to construction of Karhunen-Loéve expansion, we have $E[\big(X_{n+l}(t_i)-X_n(t_i)\big)X_n(t_j)]=0$. As a result, we have $E[X(t_i)X_n(t_j)]=R_{X_n}(t_i,t_j)$. Similarly, $E[X_n(t_i)X(t_j)]=R_{X_n}(t_i,t_j)$. Then we proceed with the computation as

$$\begin{aligned}E[|Y_n-Y|^2] &= \lim_{\min(m,k)\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{k}\big(R_{X_n}(t_i,t_j)+R_X(t_i,t_j)-2R_{X_n}(t_i,t_j)\big)(t_i-t_{i-1})(t_j-t_{j-1})\\ &= \lim_{\min(m,k)\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{k}\big(R_X(t_i,t_j)-R_{X_n}(t_i,t_j)\big)(t_i-t_{i-1})(t_j-t_{j-1})\\ &= \int_0^T\int_0^T R_X(u,s)-R_{X_n}(u,s)duds\end{aligned}$$

where the integral is Riemann integral due to its definition. Since we assume both $X_n(t)$ and $X(t)$ are integrable over $[0,T]$, the Riemann integral above exists due to Theorem 9.2.1, P. 237, [1]. Then we have

$$\begin{aligned}\lim_{n\to\infty}E[|Y_n-Y|^2] &= \lim_{n\to\infty}\int_0^T\int_0^T R_X(u,s)-R_{X_n}(u,s)duds\\ &= \int_0^T\int_0^T R_X(u,s)duds-\lim_{n\to\infty}\int_0^T\int_0^T R_{X_n}(u,s)duds.\end{aligned}$$

Due to Mercer's theorem, Theorem 7.2.2, P. 249, [2], $R_{X_n}(u,s)$ converges uniformly in u, s to $R_X(u,s)$. Then due to Theorem 2.8, P. 75, [3], we know that the above limit is well defined and we have that

$$\begin{aligned}\lim_{n\to\infty}E[|Y_n-Y|^2] &= \int_0^T\int_0^T R_X(u,s)duds-\lim_{n\to\infty}\int_0^T\int_0^T R_{X_n}(u,s)duds\\ &= \int_0^T\int_0^T R_X(u,s)duds-\int_0^T\int_0^T \lim_{n\to\infty}R_{X_n}(u,s)duds\\ &= \int_0^T\int_0^T R_X(u,s)-\lim_{n\to\infty}R_{X_n}(u,s)duds\\ &= \int_0^T\int_0^T R_X(u,s)-R_X(u,s)duds\end{aligned}$$

which concludes the proof.

References

[1] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.

[2] Bruce Hajek, *Random processes for engineers*, Cambridge University Press, 2015.

[3] John McDonald and Neil A. Weiss, *A course in real analysis*, 2nd Edition, 2012.