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Monotonicity of Means of Random Variables

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1 Problem statement

Denote a probability space as $(\Omega, \mathscr{F}, \mathbf{P})$ and two random variables (r.v.'s) defined upon it as X and Y where $X \leq Y$ holds $\forall \omega \in \Omega$. Then it holds for any type of r.v.'s that

$$E[X] \leq E[Y] \tag{1.1}$$

Prove above result by its nature, namely considering the fact that the expectation is defined via Lebesgue integration.

2 Elaboration

We only prove the case when both X and Y are non-negative r.v.'s. Here we prove three different cases.

1. First, we prove the case when both X and Y are simple. Then the proof is given in [Hand Note 1].
2. Second, if X is simple while Y is not, then suppose $\{X(\omega)|\omega \in \Omega\} = \{x_1, ..., x_m\}$. We can partition Ω into m parts depending on x_i and use the result in [Hand Note 4] to prove the monotonicity.
3. Third, if both X and Y are not simple, define

$$X_n(\omega) = \begin{cases} \frac{k-1}{2^n}, & \frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n}, \, k = 1, 2, ..., n2^n \\ n, & X(\omega) \geq n. \end{cases} \tag{2.1}$$

Similarly, define $Y_n(\omega)$. Since $X \leq Y \ \forall \omega \in \Omega$, we have $X \leq Y \ \forall \omega \in \Omega$. Then by [Hand Note 1], we have

$$E[X_n] \leq E[Y_n]. \tag{2.2}$$

By Theorem 4.2 (Consistency), P.50, [2], we know $\lim_{n\to\infty} E[X_n] = E[X]$ and $\lim_{n\to\infty} E[y_n] = E[Y]$. Take limits on both sides of Eq. (2.2), we get the result.

Remark. *When $X < Y$, the monitonicity also carry to the expectation, namely*

$$E[X] < E[Y]. \tag{2.3}$$

Denoting $Z = Y - X$. Then $Z > 0 \ \forall \omega \in \Omega$. Then $E[Z] > 0$ can be shown as a result of [Hand Note 4] claim, and this strict monotonicity is proved in step 3.

References

[1] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.

[2] Allan Gut, *Probability: A graduate course*, 2nd Edition, Springer & Verlag, 2012.

[3] Bruce Hajek, *Random processes for engineers*, Cambridge University Press, 2015.