KTH, SCHOOL OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

A Class of Equivalence Relation for the Almost Sure Convergence

Yuchao Li

October 4, 2018

1 PROBLEM STATEMENT

Probability space is given as $(\Omega, \mathscr{F}, \mathbf{P})$. Suppose a sequence of random variables (r.v.'s) defined on the probability space as $X_1, X_2, \ldots$ converge almost surely to X which is defined on the same probability space, then we have $\mathbf{P}(C) = 1$ where C is defined as

$$C = \Big\{\omega \in \Omega | \lim_{n\to\infty} |X_n(\omega) - X(\omega)| = 0\Big\}. \tag{1.1}$$

Prove the following equivalent class:

$$\mathbf{P}(C) = 1 \overset{1}{\Longleftrightarrow} \mathbf{P}(C^c) = 0 \overset{2}{\Longleftrightarrow} \mathbf{P}(\cup_{k=1}^{\infty} A(\frac{1}{k})) = 0 \overset{3}{\Longleftrightarrow} \mathbf{P}(A(\frac{1}{k})) = 0, \forall k \overset{4}{\Longleftrightarrow} \mathbf{P}(A(\epsilon)) = 0, \forall \epsilon > 0$$

$$\overset{5}{\Longleftrightarrow} \lim_{m\to\infty} \mathbf{P}(B_m(\epsilon)) = 0, \forall \epsilon > 0 \overset{6}{\Longleftrightarrow} \lim_{m\to\infty} \mathbf{P}(\cup_{n\geq m} A_n(\epsilon)) = 0, \forall \epsilon > 0$$

where $A_n(\epsilon) = \{\omega \in \Omega || X_n(\omega) - X(\omega)| > \epsilon\}$, $B_m(\epsilon) = \cup_{n\geq m} A_n(\epsilon)$ and $A(\epsilon) = \cap_{m=1}^{\infty} B_m(\epsilon)$.

2 ELABORATION

We proceed step by step.

1. The first equivalence is directly given by the axioms of probability measure P. 19 [1]. In fact, in general measure theory where the measure is denoted by μ, to state some property holds almost everywhere, which in this case is converge almost everywhere (almost surely), it is defined to be $\mu(C^c) = 0$ where set C is where the condition holds, as seen in Definition 5.2, P. 148 [2].

2. We know by P. 180 [1] that C defined in Eq. (1.1) can be rewritten as

$$C = \cap_{k=1}^{\infty} \cup_{m}^{\infty} \cap_{n \geq m} \{\omega \in \Omega | |X_n(\omega) - X(\omega)| \leq \frac{1}{k}\}.$$

Then we have

$$C^c = \cup_{k=1}^{\infty} \cap_{m}^{\infty} \cup_{n \geq m} \{\omega \in \Omega | |X_n(\omega) - X(\omega)| > \frac{1}{k}\}.$$

If we plugging in the definition of the sets $A(\epsilon)$, $B_m(\epsilon)$ and $A_n(\epsilon)$, we can rewrite the above equation as

$$C^c = \cup_{k=1}^{\infty} \cap_{m}^{\infty} \cup_{n \geq m} A_n(\frac{1}{k}), \tag{2.1}$$

$$C^c = \cup_{k=1}^{\infty} \cap_{m}^{\infty} B_m(\frac{1}{k}), \tag{2.2}$$

$$C^c = \cup_{k=1}^{\infty} A(\frac{1}{k}). \tag{2.3}$$

Then by the set equality given by Eq. (2.3), we have the second equivalence relation holds.

3. The relation

$$\mathbf{P}(\cup_{k=1}^{\infty} A(\frac{1}{k})) \implies \mathbf{P}(A(\frac{1}{k})) = 0, \forall k$$

is fairly obvious since $A(\frac{1}{k}) \subseteq \cup_{k=1}^{\infty} A(\frac{1}{k})$.

As for the inverse direction, namely

$$\mathbf{P}(A(\frac{1}{k})) = 0, \forall k \implies \mathbf{P}(\cup_{k=1}^{\infty} A(\frac{1}{k})),$$

we know by [Note 18] that for a sequence of events $(E_k)_{k=1}^{\infty}$, the following property holds

$$\mathbf{P}(\cup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} \mathbf{P}(E_k). \tag{2.4}$$

Then denote $A(\frac{1}{k}) = E_k$, we have

$$\mathbf{P}(\cup_{k=1}^{\infty} A(\frac{1}{k})) \leq \sum_{k=1}^{\infty} \mathbf{P}(A(\frac{1}{k})).$$

Since $\mathbf{P}(A(\frac{1}{k})) = 0, \forall k$, by formal definition of infinite series and the definition of its convergence and rule for assigning value for it, refer to Definition 7.2.1 and Definition 7.2.2, P. 164 [3], we know that

$$\sum_{k=1}^{\infty} \mathbf{P}(A(\frac{1}{k})) = \lim_{N \to \infty} \big(\sum_{k=1}^{N} \mathbf{P}(A(\frac{1}{k})) \big).$$

Since $\mathbf{P}(A(\frac{1}{k})) = 0 \, \forall k$, then $\sum_{k=1}^{N} \mathbf{P}(A(\frac{1}{k})) = 0 \ \forall N$. Then their limit must be 0, namely $\lim_{N \to \infty} \big(\sum_{k=1}^{N} \mathbf{P}(A(\frac{1}{k})) \big) = 0$. Therefore we have $\sum_{k=1}^{\infty} \mathbf{P}(A(\frac{1}{k})) = 0$. As a result, the implication in this direction is shown.

4. By the definition of $A(\epsilon)$, we know if $\epsilon_1 > \epsilon_2$, then $A(\epsilon_1) \subseteq A(\epsilon_2)$ and as a result $\mathbf{P}(A(\epsilon_1)) \leq \mathbf{P}(A(\epsilon_2))$. Then $\forall \epsilon$, $\exists k_\epsilon$ such that $\epsilon > \frac{1}{k_\epsilon}$. Conversely, $\forall k$, $\exists \epsilon_k$ such that $\frac{1}{k} > \epsilon_k$. By this relation, the equivalence can be established.

5. By the definition $B_m(\epsilon) = \cup_{n\geq m} A_n(\epsilon)$, we know that $B_1(\epsilon) \supseteq B_2(\epsilon) \supseteq ....$ Then by $A(\epsilon) = \cap_{m=1}^{\infty} B_m(\epsilon)$, due to Theorem 1.4.9, P. 22 [1], this equivalence relation is established. Note that since we apply Theorem 1.4.9, then we know this equivalence relation holds only for finite measure, and indeed $\mathbf{P}$ is a finite measure.

6. The equivalence relation is due to the definition of $B_m(\epsilon)$.

References

[1] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.

[2] John McDonald and Neil A. Weiss, *A course in real analysis*, 2nd Edition, 2012.

[3] Terence Tao, *Analysis I*, Springer, 2006.