KTH, SCHOOL OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

Monotonicity of Means of Simple Random Variables

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1 PROBLEM STATEMENT

Probability space is given as $(\Omega, \mathscr{F}, \mathbf{P})$. For two simple random variables X and Y, if $X(\omega) \leq Y(\omega)\ \forall \omega \in \Omega$, prove that $E[X] \leq E[Y]$.

2 ELABORATION

Since X and Y are simple random variables, we assume that X takes values from $x_1 < x_2 < ... < x_m$. As a result, Ω has a partition (A_i) where $A_i = \{\omega \in \Omega : X(\omega) = x_i\}$. Simularly, we define $y_1 < y_2 < ... < y_n$ and (B_i). Then we mix (x_i) and (y_i) together and order them as

$$x_1 < ... < x_{l_1} \leq y_1 < ... < y_{p_1} < x_{l_1+1} < ... < x_{l_2} \leq y_{p_1+1} < ... < y_{p_2} < x_{l_2+1} < (\text{or } \leq) ... < (\text{or } \leq) y_n. \tag{2.1}$$

where $l_k = m$ and $p_k = n$ and k is the number that (x_i) and (y_i) are grouped into due to the ordering and the number of groups of (x_i) and (y_i) must be the same (since the line starts with x_1 and ends with y_n). Note that between x_i and x_j $(i < j)$, it should always be $<$. The same for y_i and y_j. Between x_a and y_b (x_a in front), it should be $\leq$ and between y_b and x_a (y_b in front), it should be $<$. This means if $x_a = y_b$, we require x_a in the front. The reason would be obvious later.
Due to (1) $X(\omega) \leq Y(\omega)\ \forall \omega \in \Omega$, and (2) when $x_a = y_b$, we require x_a in the front, we have

$$\cup_{i=1}^{p_q} B_i \subseteq \cup_{i=1}^{l_q} A_i,\ q = 1, 2, ..., k. \tag{2.2}$$

Define $A_j^{(1)} = A_j \cap (\cup_{i=1}^{p_1} B_i)$ for $j = 1, ..., l_1$. Then we have

$$\cup_{i=1}^{l_1} A_i^{(1)} = \cup_{i=1}^{p_1} B_i \tag{2.3}$$

and as a result, we have

$$\sum_{i=1}^{l_1} x_i \mathbf{P}(\{\omega \in A_i^{(1)}\}) \leq y_1 \sum_{i=1}^{l_1} \mathbf{P}(\{\omega \in A_i^{(1)}\}) = y_1 \sum_{i=1}^{p_1} \mathbf{P}(\{\omega \in B_i\}) < \sum_{i=1}^{p_1} y_i \mathbf{P}(\{\omega \in B_i\}). \tag{2.4}$$

Similarly, we define $A_j^{(2)} = A_j \cap (\cup_{i=p_1+1}^{p_2} B_i)$ for $j = 1, ..., l_2$. Then we have

$$\cup_{i=1}^{l_2} A_i^{(2)} = \cup_{i=l_1+1}^{p_2} B_i \tag{2.5}$$

and as a result, we have

$$\sum_{i=1}^{l_2} x_i \mathbf{P}(\{\omega \in A_i^{(2)}\}) \leq y_{p_1+1} \sum_{i=1}^{l_2} \mathbf{P}(\{\omega \in A_i^{(2)}\}) = y_{p_1+1} \sum_{i=p_1+1}^{p_2} \mathbf{P}(\{\omega \in B_i\}) < \sum_{i=p_1+1}^{p_2} y_i \mathbf{P}(\{\omega \in B_i\}). \tag{2.6}$$

Continue k steps, we would get k inequalities like Eqs. (2.4) and (2.6). Sum them up, we get the result. Note that $E[X] = E[Y]$ iff $m = n$ and $X(\omega) = Y(\omega)\ \forall \omega \in \Omega$.

REFERENCES

[1] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.