KTH, School of Electrical Engineering and Computer Science

Topologies, metrics and standard spaces

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Question 1

Q: Given two topological spaces $(\Omega, \mathscr{T})$ and $(\Gamma, \mathscr{S})$, the two spaces are said to be homeomorphic if there is a function $f : \Omega \to \Gamma$ such that: f is one-to-one; f is continuous; f^{-1} is continuous. Prove that the $(0,1)$ and $\mathbb{R}$ are homeomorphic.
A: $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$; define $g : (0,1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ as $g(x) = -\frac{\pi}{2} + \pi x$, then $f = \tan \circ g$.

Question 2

Q: Given two topological spaces $(\Omega, \mathscr{T})$ and $(\Gamma, \mathscr{S})$, and a function $f : \Omega \to \Gamma$, let χ be the set of all sequences that converges in $(\Omega, \mathscr{T})$. Prove that

1. f is continuous $\Rightarrow \lim_n f(x_n) \to f(\lim_n x)$ holds $\forall \{x_n\} \in \chi$;
2. If $\{\Omega, \mathscr{T}\}$ is metrizable, then f is continuous $\Leftrightarrow \lim_n f(x_n) \to f(\lim_n x)$ holds $\forall \{x_n\} \in \chi$.

A: Recall the definition of f being continuous: $\forall O \in \mathscr{S}$, it holds that $f^{-1}(O) \in \mathscr{T}$. As for $\{x_n\}$ where $x_n \in \Omega$ being convergent, denote $x = \lim_n x_n$, for each $O \in \mathscr{T}$ where $x \in O$, there is an N such that $x_n \in O$ for all $n > N$.

1. $\forall O \in \mathscr{S}$ where $f(x) \in O$, since f is continuous, then we have $f^{-1}(O) \in \mathscr{T}$, besides, according to the definition of preimage, we also have $x \in f^{-1}(O)$. Since $\{x_n\}$ is convergent in $(\Omega, \mathscr{T})$, then $\exists N$ such that $x_n \in f^{-1}(O)$. Namely, with the same N, $f(x_n) \in O$ holds $\forall n > N$. Since O is arbitary open set in $(\Gamma, \mathscr{S})$, we proved that $\{f(x_n)\}$ is a convergent sequence in $(\Gamma, \mathscr{S})$ and it converges to $f(x)$.

2. The only if part has been proved in the previous step. For the if part, denote $f(\Omega) = \mathscr{R}f$, then $\forall O \in \mathscr{S}$ where $O \cap \mathscr{R}f = \emptyset$, $f^{-1}(O) = \emptyset \in \mathscr{T}$. When $O \in \mathscr{S}$ where $O \cap \mathscr{R}f \neq \emptyset$, if $f^{-1}(O) \notin \mathscr{T}$, since $(\Omega, \mathscr{T})$ is metrizable, then $\exists x \in f^{-1}(O)$ such that $B_r(x) \nsubseteq f^{-1}(O)$ $\forall r > 0$. Namely we have $B_r(x) \setminus f^{-1}(O) \neq \emptyset$ $\forall r > 0$. Construct $\{x_n\}$ such that $x_n \in B_{\frac{1}{n}} \setminus f^{-1}(O)$. Then by construction we have $x_n \notin f^{-1}(O)$ $\forall n$ and $\lim_n x = x$. However, $\lim_n f(x) = f(\lim_n x)$, namely $\exists N$ such that $f(x_n) \in O$ $\forall n > N$, or equivalently, $x_n \in f^{-1}(O)$ $\forall n > N$. Therefore, the assumption is false, and $f^{-1}(O)$ is open.

References

[1] John McDonald and Neil A. Weiss, *A course in real analysis*, 2nd Edition, 2012.

[2] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.