KTH, School of Electrical Engineering and Computer Science

Differentiation and Radon–Nikodym theorem

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Question 1

Q: Define and explain the concepts of total variation and bounded variation.
A: Let f be a real-valued function on interval $[a,b]$, the total variation of f over $[a,b]$, denoted as $V_a^b f$, is defined as

$$V_a^b f = \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\right\}$$

where the supremum is taken over all partition of the interval $[a,b]$, refer to [1] P. 117. For definition of partition of an interval, refer to [2] Definition 11.1.10, P. 269.

Question 2

Q: Define and explain the concepts of an absolutely continuous real-valued function.
A: Suppose that f is defined on $\mathbb{R}$, f' exists almost everywhere and is Lebesgue integrable on $\mathbb{R}$, and

$$f(x) = \int_{\infty}^{x} f'(t)dt, \quad -\infty < x < \infty.$$

Then f is absolutely continuous on $\mathbb{R}$. This is [3] Definition 8.8, P. 293.

Question 3

Q: Define and explain the concepts of discrete, continuous and absolutely continuous random variables.

A: Given a probability space $(\Omega, \mathscr{A}, \mathbf{P})$ and a random variable X defined on it, the random variable is called

- discrete, if there is a countable set $K \in \mathbb{B}$ such that $\mathbf{P}(X \in K) = 1$. In fact, there exists a countable $K_0 \neq \emptyset$ such that $\mathbf{P}(X \in K_0) = 1$, $\mathbf{P}(X = x_i) > 0$ holds $\forall x_i \in K_0$, and $\forall K$ where $\mathbf{P}(X \in K) = 1$, it holds that $K_0 \subseteq K$. Refer to [4] Lemma 2.5.2, P. 72.
- continuous, if $\mathbf{P}(X = x) = 0$ holds for all $x \in \mathbb{R}$.
- absolutely continuous, if there is a nonnegative $\mathscr{B}-$measurable function f_X such that

$$\mu_X(B) = \int_B f_X(x)dx$$

  holds for all $B \in \mathbb{B}$ where μ_X is the distribution of X.

QUESTION 4

Q: Define the Radon-Nikodym derivative and relate it to the concept of an absolutely continuous random variable.
A: If μ and ν are $\sigma-$finite on $(\Omega, \mathscr{A})$ and $\nu \ll \mu$, then there is a nonnegative extended real-valued $\mathscr{A}-$measurable function f on Ω such that

$$\nu(A) = \int_A f d\mu$$

for any $A \in \mathscr{A}$. Further more, f is unique $\mu-$a.e., and is named as Radon-Nikodym derivative of ν w.r.t. μ, denoted as

$$f = \frac{d\nu}{d\mu}.$$

QUESTION 5

Q: Assume that $\{f_n\}$ is a sequence of functions that converges pointwise to $f < \infty$. Prove that $V_a^b f \leq \liminf_n V_a^b f_n$.
A: Since $\{f_n\}$ is the sequence of function that converges pointwise $f < \infty$ on $[a, b]$, then for any $x \in [a, b]$, we have $\lim_{n\to\infty} f_n(x) = f(x) < \infty$. With conventional definition of a sequence being convergent, we know $\{f_n\}$ are real-valued functions on $[a, b]$. Otherwise we need to assume that $\{f_n\}$ are real-valued functions on $[a, b]$ such that $V_a^b f_n$ are well-defined. Then we have for any partition of $[a, b]$ given by $a = x_0 < x_1 < ... < x_{m-1} < x_m = b$, we have

$$\begin{aligned}
& f(x_k) - f(x_{k-1}) = \lim_{n\to\infty} f_n(x_k) - f_n(x_{k-1}) \implies f(x_k) - f(x_{k-1}) = \liminf_{n\to\infty} f_n(x_k) - f_n(x_{k-1}) \\
& \implies |f(x_k) - f(x_{k-1})| = \liminf_{n\to\infty} |f_n(x_k) - f_n(x_{k-1})| \\
& \implies \sum_{k=1}^{m} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{m} \liminf_{n\to\infty} |f_n(x_k) - f_n(x_{k-1})|.
\end{aligned}$$

By sum rule of limit, we have

$$\sum_{k=1}^{m} |f(x_k) - f(x_{k-1})| = \liminf_{n\to\infty} \sum_{k=1}^{m} |f_n(x_k) - f_n(x_{k-1})|.$$

By definition of total variation, we have $\sum_{k=1}^{m} |f_n(x_k) - f_n(x_{k-1})| \le V_a^b f_n$, as a result, we have

$$\sum_{k=1}^{m} |f(x_k) - f(x_{k-1})| \le \liminf_{n\to\infty} V_a^b f_n. \tag{0.1}$$

Since 0.1 holds for any partition on $[a, b]$, then it holds that

$$V_a^b f = \sup\left\{ \sum_{k=1}^{m} |f(x_k) - f(x_{k-1})| \right\} \le \liminf_{n\to\infty} V_a^b f_n.$$

QUESTION 6

Q: Give an example of a function f that is absolutely continuous on $[0,1]$ but is such that f' is not Riemann integrable on $[0,1]$. That is, for such a function

$$f(x) = f(0) + \int_0^x f'(x)\,dt$$

does not hold when the integral is Riemann integral.
A: Define function $g : [0,1] \to \{0,1\}$ to be

$$g(x) = \begin{cases} 0, & x \in [0,1] \cap \mathbb{Q}, \\ 1, & \text{o.w.}. \end{cases}$$

Then it can be shown that $g \in \mathscr{L}^1([0,1])$ since we have

$$\int_{[0,x]} g(t)\,dt = \int_{[0,x]\setminus\mathbb{Q}} g(t)\,dt + \int_{[0,x]\cap\mathbb{Q}} g(t)\,dt = \int_{[0,x]\setminus\mathbb{Q}} g(t)\,dt + 0 = x; \int_{\{0\}} g(t)\,dt = 0.$$

holds for $x \in (0,1]$. Then we define $f : [0,1] \to \mathbb{R}$ as

$$f(x) = \int_{[0,x]} g(t)\,dt.$$

Then by First fundamental theorem of calculus for Lebesgue integral, [3] Theorem 8.5, P. 289, it holds that $f' = g$ λ-a.e. on $[0,1]$ yet g is not Riemann integrable.

QUESTION 7

Q: Let X be a discrete random variable on $(\Omega, \mathscr{A}, \mathbf{P})$ with probability mass function $p_X(x) = \mathbf{P}(\{\omega : X = x\})$. Let μ be a counting measure on $(\mathbb{R}, \mathscr{B})$, that is $\mu(B) = |B|$ (cardinality) for $B \in \mathscr{B}$. Show that $\mu_X \ll \mu$ and that

$$p_X = \frac{d\mu_X}{d\mu}.$$

A: Since X is a discrete random variable on $(\Omega, \mathscr{A}, \mathbf{P})$, then there is a countable set $K \in \mathscr{B}$ such that $\mathbf{P}(X \in K) = 1$. For counting measure μ on $(\mathbb{R}, \mathscr{B})$, the only measure zero set is the empty set, namely $\mu(\emptyset) = 0$. Since $\mu_X(\emptyset) = \mu_X(X \in \mathbb{R}^c) = 0$, so we have $\mu_X \ll \mu$.
Introduce measure space $(K, \mathscr{B}_{|K}, \mu_{X|K})$ and $(K, \mathscr{B}_{|K}, \mu_{|K})$, then apparently we have $\mu_{X|K} \ll \mu_{|K}$, and both are σ-finite since K is countable. Then by Radon-Nikodym theorem, there exists a nonnegative $\mathscr{B}_{|K}$-measurable extended real-valued function, denoted as $\frac{d\mu_{X|K}}{d\mu_{|K}}$ such that it holds

$$\mu_{X|K}(B) = \int_B \frac{d\mu_{X|K}}{d\mu_{|K}} d\mu_{|K}.$$

Since K is countable, then $B \in \mathscr{B}_{|K}$ is countable, then we have by definition that

$$\mu_{X|K}(B) = \mathbf{P}(X \in B) = \sum_{x_i \in B} p_X(x_i) = \sum_{x_i \in B} p_X(x_i) = \sum_{x_i \in B} p_X(x_i)\mu_{|K}(\{x_i\}) = \int_B p_X(x_i)\mu_{|K}.$$

Therefore, we have that

$$p_X(x_i) = \frac{d\mu_{X|K}}{d\mu_{|K}} \quad \mu_{|K} - \text{a.e..}$$

However, since $\mu_{|K}$ is the counting measure, we have it holds everywhere in $(K, \mathscr{B}_{|K})$. In fact, we can show the above result holds for measure space $(\mathbb{R}, \mathscr{B})$. That is $\forall A \in \mathscr{B}$, we have

$$\mu_X(A) = \mu_X(A \cap K) + \mu_X(A \setminus K) = \mu_{X|K}(A \cap K) + 0 = \mu_{X|K}(A \cap K)$$

and since $\mathbf{P}(X \notin K) = 0$, then we have

$$\mu_X(A \setminus K) = 0 = 0 \int_{A \setminus K} d\mu = \int_{A \setminus K} p_X(x) d\mu.$$

References

[1] Elias M. Stein and Rami Shakarchi, *Real analysis: Measure theory, integration, and Hilbert spaces*, Princeton University Press, 2009.

[2] Terence Tao, *Analysis I*, Springer, 2006.

[3] John McDonald and Neil A. Weiss, *A course in real analysis*, 2nd Edition, 2012.

[4] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.