KTH, SCHOOL OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

General integration theory

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QUESTION 1

Q: Given a finite measure space $(\Omega, \mathscr{A}, \mu)$ and a real-valued measurable function $f : \Omega \to [0,1]$, prove that

$$\lim_{n\to\infty} \int f^{1/n} d\mu = \mu\big(f^{-1}((0,1])\big),$$

and

$$\lim_{n\to\infty} \int f^{n} d\mu = \mu\big(f^{-1}(\{1\})\big).$$

A: Given a measure space $(\Omega, \mathscr{A}, \mu)$ and a real-valued measurable function $f : \Omega \to [0, 1]$, denote $g_n = f^{1/n}$. We also define constant function $l = 1$. It can be shown that $g_n \le l$. Since μ is finite measure, then l is integrable. Therefore, we can apply DCT. Denote $\lim g_n = g$. Then we have $g_n \uparrow g$. Then we have for $\omega \in f^{-1}((0,1])$

$$\ln g_n \le \ln g \iff \frac{1}{n} \ln f \le \ln g \implies \sup(\frac{1}{n} \ln f) \le \ln g \implies 0 \le \ln g.$$

Due to the definition of supremum, we have $g = 1$. For $\omega \in f^{-1}(\{0\})$, $g = 0$. So we know that g is a simple function. By DCT, we have

$$\begin{aligned}
\lim_{n\to\infty} \int f^{1/n} d\mu &= \lim_{n\to\infty} \int g_n d\mu \\
&= \int \lim_{n\to\infty} g_n d\mu \\
&= \int g d\mu \\
&= 1 \cdot \mu\big(f^{-1}((0,1])\big) + 0 \cdot \mu\big(f^{-1}(\{0\})\big) \\
&= \mu\big(f^{-1}((0,1])\big).
\end{aligned}$$

Similarly, denote $h_n = f^n$. Again, we can directly see that h_n is bounded by l and we can apply DCT. Besides, h_n converge to h which is defined as $h = 1$ for $\omega \in f^{-1}(\{1\})$ and $h = 0$ o.w.. By apply DCT, we have

$$\begin{aligned}
\lim_{n\to\infty} \int f^n d\mu &= \lim_{n\to\infty} \int h_n d\mu \\
&= \int h d\mu \\
&= 0 \cdot \mu\big(f^{-1}([0,1))\big) + 1 \cdot \mu\big(f^{-1}(\{1\})\big) \\
&= \mu\big(f^{-1}(\{1\})\big).
\end{aligned}$$

REFERENCES

[1] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.