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General measure theory

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Question 1

Q: Define/motivate the concepts σ-algebra, measure, measure space, measurable space and measurable set.
A: Given universal set Ω, a σ-algebra $\mathscr{F}$ of Ω is a set of subsets of Ω such that 1) $\Omega \in \mathscr{F}$; 2) For $A \in \Omega$, $A \in \mathscr{F} \implies A^c \in \mathscr{F}$; 3) For $\{A_i\}_{i=1}^{\infty}$, $A_i \in \Omega \, \forall i \implies \cup_{i=1}^{\infty} A_i \in \mathscr{F}$. Measure is given as μ, measure space given as $(\Omega, \mathscr{F}, \mu)$. Measurable space given as $(\Omega, \mathscr{F})$. Measurable set is $A \subset \Omega$ such that $A \in \mathscr{F}$.

Question 2

Q: On the real line, define/motivate Borel measurable set and Borel measurable function. Discuss the relation between Borel measurable and continuous functions.
A: Denote $\mathscr{O}$ as the set of open sets of some universal set. Then we have $\mathscr{B} = \sigma(\mathscr{O})$. $B \in \mathscr{B}$ is called Borel-measurable. For $f : \mathscr{U} \to \mathscr{V}$, it is Borel-measurable if $f^{-1}(O)$ is Borel measurable interms of the Borel σ algebra of $\mathscr{U}$.

Question 3

Q: Define and discuss the concepts almost everywhere (a.e.), convergence a.e., and convergence in measure. For a finite measure, prove that convergence a.e. implies convergence in measure. Describe a counterexample showing that the reverse statement does not hold in general.

A: First of all, for measure space $(\Omega, \mathscr{F}, \mu)$ where $\mu(\Omega) < \infty$, and $A_n \downarrow A$, we have

$$\lim_{n\to\infty} \mu(A_n) = \mu(\cap_{n=1}^{\infty} A_n). \tag{0.1}$$

Refer to [1] Theorem 1.4.9 to check how the finiteness of the measure play a role.
Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence a measurable function defined on the measure space and converge almost everywhere (a.e.), then we have $\mu(\lim_{n\to\infty} f_n = f) = \mu(\Omega)$. Suppose $\{f_n\}$ does not converge in measure to f, that is, $\exists\varepsilon$, $\exists\delta$ and $\exists\{f_{n_k}\}_{k=1}^{\infty}$, such that $\mu(|f_{n_k} - f| \geq \varepsilon) > \delta$ holds $\forall k$. However, if we denote $B_k = \{|f_{n_k} - f| \geq \varepsilon\}$, then due to finiteness of measure, we have

$$\mu(\limsup_{k\to\infty} B_k) = \mu(\cap_{m=1}^{\infty} \cup_{k=m}^{\infty} B_k) = \lim_{m\to\infty} \mu(\cup_{k=m}^{\infty} B_k) \geq \delta. \tag{0.2}$$

That is, there exists a sub-sequence of $\{f_n\}_{n=1}^{\infty}$ that does not converge to f a.e., which contradicts the assumption.

Question 4

Q: Given a measure space $(\Omega, \mathscr{A}, \mu)$, let $\{E_n\}$ be an infinite sequence of $\mathscr{A}$-measurable sets, with $\mu(\cup_n E_n) < \infty$. Prove that

$$\mu(\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} E_k) \leq \liminf_{n\to\infty} \mu(E_n) \leq \limsup_{n\to\infty} \mu(E_n) \leq \mu(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k).$$

A: We only prove the last inequality. Given $(\Omega, \mathscr{A}, \mu)$, and $\mu(\Omega) < \infty$. Suppose $\{E_n\}_{n=1}^{\infty}$ is a sequence of $\mathscr{A}$-measurable sets. Denote $F_n = \cup_{k=n} E_k$, then $F_n \downarrow F$ where $F = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$. Due to finiteness of measure μ, we have

$$\mu(F) = \mu(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k) = \lim_{n\to\infty} \mu(\cup_{k=n}^{\infty} E_k) \tag{0.3}$$

and we know that $\mu(\cup_{k=n}^{\infty} E_k) \downarrow$ as $n \uparrow$. As a result, we have

$$\sup_{m\geq n} \mu(\cup_{k=m}^{\infty} E_k) = \mu(\cup_{k=n}^{\infty} E_k). \tag{0.4}$$

Since $\mu(E_m) \leq \mu(\cup_{k=m}^{\infty} E_k)$, take supreme on both sides, we have

$$\sup_{m\geq n} \mu(E_m) \leq \sup_{m\geq n} \mu(\cup_{k=m}^{\infty} E_k) = \mu(\cup_{k=n}^{\infty} E_k).$$

Take limits on both sides, we have

$$\limsup_{n\to\infty} \mu(E_n) \leq \lim_{n\to\infty} \mu(\cup_{k=n}^{\infty} E_k) = \mu(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k).$$

References

[1] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.