KTH, School of Electrical Engineering and Computer Science

The Lebesgue integral on the real line

Yuchao Li

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Question 1

Q: Define 'simple function' and prove that if f is a nonnegative Lebesgue measurable function, then there is a nondecreasing sequence of simple nonnegative functions that converges pointwise to f. Also prove the converse, i.e., that the pointwise limit of a sequence of simple nonnegative functions is Lebesgue measurable.

A:

1. $f : \mathbf{R} \to \mathbf{R}$ is simple if f is Lebesgue measurable and $f(x) = \sum_{i\in\mathscr{I}} \chi_{A_i}(x) f_i$ where $\{A_i\}_{i\in\mathscr{I}}$ is a partition of $\mathbf{R}$ and $|\mathscr{I}| < \infty$.

2. For nonnegative Lebesgue measurable function $f : \mathbf{R} \to \mathbf{R}_+ \cup \{0\}$, we define

$$f_n(x) = \begin{cases} \frac{k-1}{2^n}, & \frac{k-1}{2^n} \le f(x) < \frac{k}{2^n},\ k = 1, 2, \ldots, n2^n, \\ n, & x \ge n. \end{cases} \tag{0.1}$$

   It can be shown that f_n is simple, that is all $f^{-1}(E_{n,k})$ is Lebesgue measurable where $E_{n,k} = [\frac{k-1}{2^n}, \frac{k}{2^n})$ for $k = 1, 2, \ldots, n2^n$ and $E_{n,n2^n+1} = [E_{n,k} = [n, \infty)$, and increasing. Besides, they are also point-wise converge to f.

3. Conversely, if $\{f_i(x)\}_{i=1}^{\infty}$ are a sequence of simple function and pointwise converge to f, then we define $g_i(x) = \inf_{m\ge i} f_m(x)$. We know that if a sequence of increasing simple function pointwise converge to a function f, then f is Lebesgue measurable. In this case, we know $\lim g_i(x) = \lim f_i(x)\ \forall x$, then then $\lim g_i(x) = f(x)$. As a result, $f(x)$ is Lebesgue measurable.

QUESTION 2

Q: Define the Lebesgue integral of a nonnegative Lebesgue measurable function. Use the result in previous problem to illustrate graphically how the value of the integral is obtained as a nondecreasing sequence of approximations to the final value (the limit).

A: First, we need to define Lebesgue integral for simple function, namely for simple function $f(x) = \sum_{i \in \mathscr{I}} \chi_{A_i}(x) f_i$, its integral is given as

$$\int f d\lambda = \sum_{i \in \mathscr{I}} f_i \lambda(A_i). \tag{0.2}$$

Then for nonsimple Lebesgue measurable function, their integral is given as the limit of sequence of integrals of simple functions given in Eq. (0.1), namely

$$\int f d\lambda = \lim_{n\to\infty} \int f_n d\lambda \tag{0.3}$$

where the right-hand side is always well-defined and increasing.

QUESTION 3

Q: Prove the MCT.

A: Since f_i measurable and $\lim_{i\to\infty} f_i(x) = f(x)$ pointwise, so we have $f(x)$ measurable. Denote $K = \int f d\lambda$. Since $\{f_i\}_{i=1}^{\infty}$ is increasing, by monotonicity preserved by integral (refer to [Hand Note 11] in SF2940 for details), we have $\{\int f_i d\lambda\}_{i=1}^{\infty}$ as a sequence is increasing. Denote $L = \lim \int f_i d\lambda$. Again, by monotonity preserved by the integral, we have $L \leq K$. Let s be some positive function such that $0 \leq s(x) \leq f(x)$ and $0 < \alpha < 1$ be some constant, then we define $E_i = \{x : f_i(x) \geq \alpha s(x)\}$. Then we have $E_i \uparrow$ and $\cup_i E_i = \mathbf{R}$. To see this, notice that $E_i = \cup_i \{x : f_i(x) \geq \alpha s(x)\}$. Since $\cup_i \{x : f_i(x) \geq \alpha s(x)\} = \{x : \sup f_i(x) \geq \alpha s(x)\}$ (refer to [1] P.29 side note for details), then we have $\cup_i \{x : f_i(x) \geq \alpha s(x)\} = \{x : f(x) \geq \alpha s(x)\} = \mathbf{R}$ due to the definition of $s(x)$ and α. Then we have

$$\alpha \int s d\lambda = \lim_{n\to\infty} \int_{E_n} s d\lambda \leq \limsup_{n\to\infty} f_n d\lambda = L.$$

Namely $\int s d\lambda \leq \alpha^{-1} L$. Since it holds for any s, then we have

$$\int f d\lambda = \sup_{0 \leq s \leq f} \int s d\lambda \leq \alpha^{-1} L.$$

Since the above inequality holds for any $\alpha \in (0,1)$, take $\alpha \to 1$, we get $K \leq L$. As a result, we have $K = L$.

QUESTION 4

Q: Prove Fatou's lemma (you can use the MCT without proving it).

A: Since $\lim_{i\to\infty} f_i = f$ and Lebesgue measurable, then f is Lebesgue measurable. We define another sequence of functions as $g_i = \sup_{m\geq i} f_m$, then we have $\{g_i\}_{i=1}^{\infty}$ is increasing and Lebesgue measurable and $\lim g_i = f$. Due to MCT, we have

$$\lim_{i\to\infty}\int g_i d\lambda = \int \lim_{i\to\infty} g_i d\lambda.$$

Since $g_i \leq f_i \ \forall i$, due to monotonicity preserved by the Lebesgue integral and limit, we have

$$\liminf_{i\to\infty}\int g_i d\lambda \leq \liminf_{i\to\infty}\int f_i d\lambda.$$

Since $\lim_{i\to\infty}\int g_i d\lambda = \liminf_{i\to\infty}\int g_i d\lambda$, the proof is done.

QUESTION 5

Q: Prove the DCT (you can use Fatou's lemma without proving it).
A: Since $|f_n| \leq g$, f_n Lebesgue measurable and g Lebesgue integrable, and $\lim_{n\to\infty} f_n = f$ pointwise, then we have f_n and f Lebesgue integrable. Besides, we have $\lim_{n\to\infty}(f_n+g) = f+g$ and $\lim_{n\to\infty}(g-f_n) = g-f$. Due to Fatou's Lemma, we have

$$\int f d\lambda \leq \liminf_{n\to\infty}\int f_n\lambda. \tag{0.4}$$

Besides, due to Fatou's Lemma, we also have

$$\int g - f d\lambda \leq \liminf_{n\to\infty}\int g - f_n d\lambda.$$

For both side, due to linearity of Lebesgue integral, we have

$$\int g d\lambda - \int f d\lambda \leq \liminf_{n\to\infty}(\int g d\lambda - \int f_n d\lambda)$$
$$\int g d\lambda - \int f d\lambda \leq \int g d\lambda + \liminf_{n\to\infty}(-\int f_n d\lambda)$$

Since we have $\liminf(-a_n) = -\limsup a_n$, then we have

$$\int g d\lambda - \int f d\lambda \leq \int g d\lambda + \liminf_{n\to\infty}(-\int f_n d\lambda) = \int g d\lambda - \limsup_{n\to\infty}\int f_n d\lambda.$$

So we have $\int f d\lambda \geq \limsup_{n\to\infty}\int f_n d\lambda$. Combined with Eq. (0.4), we have the proof done.

REFERENCES

[1] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.