KTH, School of Electrical Engineering and Computer Science

Basics and Lebesgue Measure on the Real Line

Yuchao Li

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Question 1

Q: For sequence $\{x_n\}$ and $\{y_n\}$,

1. define $\limsup x_n$ and $\liminf x_n$;
2. prove that
$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n;$$
3. if $\limsup y_n < \infty$ exists, prove that
$$\limsup(x_n + y_n) = \limsup x_n + \limsup y_n;$$
4. let
$$a_n = \frac{1}{n}\sum_{k=1}^{n} x_k,$$
prove that
$$\liminf x_n \leq \liminf a_n \leq \limsup a_n \leq \limsup x_n,$$
and conclude that if $\lim x_n$ exists, so does $\lim a_n$; does the converse hold?

A: For sequence $\{x_n\}$ and $\{y_n\}$,

1. For a sequence $\{x_n\}$, $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{m\geq n} x_m$. Similarly, $\liminf_{n\to\infty} x_n$ can be defined.

2. First, note that
$$\limsup_{n\to\infty}(x_n+y_n)=\lim_{n\to\infty}\big(\sup_{m\geq n}(x_m+y_m)\big).$$
Then note that
$$\limsup_{n\to\infty} x_n+\limsup_{n\to\infty} y_n=\lim_{n\to\infty}\sup_{m\geq n} x_m+\lim_{n\to\infty}\sup_{m\geq n} y_m=\lim_{n\to\infty}(\sup_{m\geq n} x_m+\sup_{m\geq n} y_m).$$
Since $\forall\, n$, we have
$$x_i+y_i\leq\sup_{m\geq n} x_m+\sup_{m\geq n} y_m\,\forall\, i\geq n,$$
then
$$\sup_{m\geq n}(x_m+y_m)\leq\sup_{m\geq n} x_m+\sup_{m\geq n} y_m\,\forall\, n, \tag{0.1}$$
due to definition of sup. Take limits on both sides of Eq. (0.1), the proof is done.

3. First of all, for any real sequence $\{y_n\}$, we can prove that
$$\limsup(-y_n)=\liminf y_n \tag{0.2}$$
Then by the Eq. (0.1), we have also
$$\limsup x_n=\limsup\big((x_n+y_n)-y_n\big)\leq\limsup(x_n+y_n)+\limsup(-y_n).$$
Then due to Eq. (0.2), we have
$$\limsup x_n\leq\limsup(x_n+y_n)-\liminf y_n\implies\limsup x_n+\liminf y_n\leq\limsup(x_n+y_n).$$
Since it also holds that
$$\limsup(x_n+y_n)\leq\limsup x_n+\limsup y_n$$
and $\lim y_n=\liminf y_n=\limsup y_n$. Then the proof is done.

4. By definition, we have $a_n=\frac{1}{n}\sum_{k=1}^{n} x_k$ and $\liminf a_n\leq\limsup a_n$.
Then we proceed to show that $\limsup a_n\leq\limsup x_n$. For any $n_0>1$, we can find and fix some $j<n_0$, such that we have
$$a_{n_0}=\frac{1}{n_0}\sum_{k=1}^{n_0} x_k=\frac{1}{n_0}\sum_{k=1}^{j} x_k+\frac{1}{n_0}\sum_{k=j+1}^{n_0} x_k\leq\frac{1}{n_0}\sum_{k=1}^{j} x_k+\frac{n_0-j}{n_0}\sup_{m\geq j} x_m\leq\frac{1}{n_0}\sum_{k=1}^{j} x_k+\sup_{m\geq j} x_m. \tag{0.3}$$
Since j is fixed, then both $\sum_{k=1}^{j} x_k$ and $\sup_{m\geq j} x_m$ are constants. Besides, it's also obvious that since $j<n_0$, then $j<n\,\forall\, n\geq n_0$. As a result, we have
$$a_n\leq\frac{1}{n}\sum_{k=1}^{j} x_k+\sup_{m\geq j} x_m,\,\forall\, n\geq n_0. \tag{0.4}$$

Take $\limsup_{n\to\infty}$ on both sides of Eq. (0.4), and since $\sum_{k=1}^{j} x_k$ and $\sup_{m\geq j} x_m$ are constants, we have

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty}\Big(\frac{1}{n}\sum_{k=1}^{j} x_k + \sup_{m\geq j} x_m\Big) = \sup_{m\geq j} x_m. \tag{0.5}$$

We can simply by setting $n_0 = j$ to show that Eq. (0.5) holds $\forall j$. Denote $c_j = \sup_{m\geq j} x_m$. Then we have $\limsup_{n\to\infty} a_n \leq c_j, \forall j$. Then we have

$$\limsup_{n\to\infty} a_n \leq \lim_{j\to\infty} c_j = \lim_{j\to\infty}\sup_{m\geq j} x_m. \tag{0.6}$$

For the proof of $\liminf x_n \leq \liminf a_n$, it's quite similar therefore we will be concise. For any n_0, fix some j where $j < n_0$, then

$$a_{n_0} = \frac{1}{n_0}\sum_{k=1}^{n_0} x_k = \frac{1}{n_0}\sum_{k=1}^{j} x_k + \frac{1}{n_0}\sum_{k=j+1}^{n_0} x_k \geq \frac{1}{n_0}\sum_{k=1}^{j} x_k + \frac{n_0 - j}{n_0}\inf_{m\geq j} x_m. \tag{0.7}$$

Since $\frac{1}{n}\sum_{k=1}^{j} x_k + \frac{n-j}{n}\inf_{m\geq j} x_m = \frac{1}{n}(\sum_{k=1}^{j} x_k - j\inf_{m\geq j} x_m) + \inf_{m\geq j} x_m$, then we have

$$\frac{1}{n}\Big(\sum_{k=1}^{j} x_k - j\inf_{m\geq j} x_m\Big) + \inf_{m\geq j} x_m \leq a_n, \forall n \geq n_0. \tag{0.8}$$

Take $\liminf_{n\to\infty}$ on both sides, we have

$$\liminf_{n\to\infty}\Big(\frac{1}{n}\Big(\sum_{k=1}^{j} x_k - j\inf_{m\geq j} x_m\Big) + \inf_{m\geq j} x_m\Big) = \inf_{m\geq j} x_m \leq \liminf_{n\to\infty} a_n. \tag{0.9}$$

Similarly, we can argue that Eq. (0.9) holds $\forall j$. Take $\lim_{j\to\infty}$, we have the proof done.

If $\lim_{n\to\infty} x_n$ exists, we have $\liminf x_n = \limsup x_n$. As a result, we have $\liminf a_n = \limsup a_n$. The converse may not be true. Counter example is given as $x_n = \frac{1+(-1)^n}{2}$.

QUESTION 2

Q: Prove that a function $f(x)$ is continuous iff $f^{-1}(O)$ is open for every open $O \subset \mathbb{R}$.
A: Only if: if $O \subset \mathbb{R}$ open and $f(x_0) \in O$, then $\exists\varepsilon$ such that $(f(x_0)-\varepsilon, f(x_0)+\varepsilon) \subset O$. Since $f(x)$ is continuous, for ε, $\exists\delta(\varepsilon)$, such that $f(x_0)-\varepsilon < f(x) < f(x_0)+\varepsilon$ holds $\forall x \in (x_0-\delta(\varepsilon), x_0+\delta(\varepsilon))$, which means $(x_0-\delta(\varepsilon), x_0+\delta(\varepsilon)) \subset f^{-1}(O)$.
If: if $x_0 \in f^{-1}(O)$, then $f(x_0) \in O$. Since for any O open, we have $f^{-1}(O)$ open, then $\forall\varepsilon > 0$, denote B as $B = (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$ which is open, we have $f^{-1}(B)$ open. Since $f(x_0) \in B$, $x_0 \in f^{-1}(B)$. B open, then $\exists\delta$ such that $(x_0-\delta, x_0+\delta) \subset f^{-1}(B)$, that is $\forall x \in (x_0-\delta, x_0+\delta)$, $f(x_0)-\varepsilon < f(x) < f(x_0)+\varepsilon$, which means the function is continuous.

Question 3

Q: Define Lebesgue outer measure λ^*, and explain what goes wrong when trying to use λ^* as a universal measure for "length" on the real line. Then define Lebesgue measure λ and motivate the definition.
A: For interval I of the form $[a, b]$, $[a, b)$, $(a, b]$ and (a, b) where $a \leq b$, define length $\jmath(I) = \ell(I) = b - a$. Besides, we know a fact that $\forall B$ open, we have $B = \cup_{i \in \mathscr{I}} I_i$ where $\mathscr{I}$ is the countable index set and I_i pairwise disjoint. Therefore, $\forall B$ open, $\jmath(\cdot)$ is defined. Note that $\forall A \subset \mathbb{R}$, there exists at least one such B which is open and $A \subset B$ since $\mathbb{R}$ is one such set, that is open and contains A. Therefore, $\{\jmath(B) : B \text{ open and } A \subset B\} \neq \emptyset$ holds $\forall A \subset \mathbb{R}$. Therefore, its infimum is well defined. Then $\forall A \subset \mathbb{R}$, we define

$$\lambda^*(A) = \inf\{\jmath(B) : B \text{ open and } A \subset B\} \tag{0.10}$$

as Lebesgue outer measure $\forall A \subset \mathbb{R}$. What goes wrong for $\lambda^*(\cdot)$ is that $\exists B, B_1, B_2$ where $B_1 \cap B_2 = \emptyset$ and $B = B_1 \cup B_2$, such that $\lambda^*(B) \neq \lambda^*(B_1) + \lambda^*(B_2)$, which is a desired property we would like measure to have.
To have that, notice $\forall O$ open, it holds that

$$\lambda^*(A) = \lambda^*(A \cap O) + \lambda^*(A \cap O^c). \tag{0.11}$$

So we define Lebesgue measurable set as

$$\mathscr{L} = \{W \in \mathbb{R} : \lambda^*(A) = \lambda^*(A \cap W) + \lambda^*(A \cap W^c), \forall A \in \mathbb{R}\}. \tag{0.12}$$

Then the Lebesgue measure is defined as $\lambda = \lambda^*_{|\mathscr{L}}$.

Question 4

Q: For a function $f(x)$, define what it means for f to be Lebesgue measurable. Based on the definition, argue that all continuous functions are Lebesgue measurable but there are Lebesgue measurable functions that are not continuous.
A: For $f : \mathbb{R} \to \mathbb{R}$, if $f^{-1}(O) \in \mathscr{L}$ holds $\forall O \in \mathbb{R}$ open, then f is Lebesgue measurable. Since $\forall A \in \mathbb{R}$ which is open, $A \in \mathscr{L}$, so continuous function is Lebesgue measurable. Since continuous function is not closed under pointwise convergent but Lebesgue measurable function does, that means, $\exists \{f_n\}$ continuous, such that $\lim_{n\to\infty} f_n(x) = f(x)$ holds $\forall x \in \mathbb{R}$ but $\lim_{n\to\infty} \|f_n(x) - f(x)\|_\infty \neq 0$.

Theorem 0.1. $\{f_n\}$ *Lebesgue measurable, and* $\lim_{n\to\infty} f_n(x) = f(x)$ *holds* $\forall x \in \mathbb{R}$*, then* f *is Lebesgue measurable.*

Proof. Use the same technique applied in Theorem 1.5.8 [1]. □

References

[1] Timo Koski, *Lecture notes: Probability and random processes at KTH*, 2017.