KTH, SCHOOL OF ELECTRICAL ENGINEERING

On Woodbury Matrix Identity

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1 PROBLEM STATEMENT

According to **Lecture Note 2**, **Lemma 2**, *Woodbury matrix identity* is given by the following equation

$$(X + ZYZ^T)^{-1} = X^{-1} - X^{-1}Z(Y^{-1} + Z^T X^{-1} Z)^{-1} Z^T X^{-1} \tag{1.1}$$

for calculating the matrix inverse of $X + ZYZ^T$. Besides, by applying Eq. (1.1), **Lecture Note 2**, **Lemma 2** also give a result that

$$Y^{-1} Z^T (X^{-1} + ZY^{-1} Z^T)^{-1} = (Y + Z^T XZ)^{-1} Z^T X. \tag{1.2}$$

Regarding the conditions needed to make those equations hold, we have the following two statements. Elaborate those statements.

1. To make Eq. (1.1) hold, we need not only X and Y invertable, but also $X + ZYZ^T$ invertable. Namely, the condition that X and Y invertable does not necessarily imply that $X + ZYZ^T$ invertable. However, X, Y and $X + ZYZ^T$ invertable does imply that $Y^{-1} + Z^T X^{-1} Z$ invertable.

2. X, Y and $X + ZYZ^T$ invertable does not imply that $Y + Z^T XZ$ invertable. Namely, the singularity of $X + ZYZ^T$ and $Y + Z^T XZ$ are independent. Therefore, to make Eq. (1.2) hold, we need precondition that X, Y and $Y + Z^T XZ$ invertable. Similarly, X, Y and $Y + Z^T XZ$ invertable does imply $X^{-1} + ZY^{-1} Z^T$ invertable.

2 Elaboration

First, we define $X = I$, $Z = I$ and $Y = -I$. Then $X + ZYZ^T$ is zero matrix, which is not invertable. Therefore, to make Eq. (1.1) hold, we need to literally require $X + ZYZ^T$ invertable. To see that X, Y and $X + ZYZ^T$ invertable does imply that $Y^{-1} + Z^T X^{-1} Z$ invertable, we follow the approach of *LDU decomposition* given by **Note 5**. Define $\mathscr{H}$ by the following equation

$$\mathscr{H} = \begin{bmatrix} X & Z \\ Z^T & -Y^{-1} \end{bmatrix}. \tag{2.1}$$

Then define $\mathscr{L}_1$ and $\mathscr{R}_1$ as

$$\mathscr{L}_1 = \begin{bmatrix} I & 0 \\ -Z^T X^{-1} & I \end{bmatrix}, \mathscr{R}_1 = \begin{bmatrix} I & -X^{-1}Z \\ 0 & I \end{bmatrix} \tag{2.2}$$

Then we have

$$\mathscr{H} = \mathscr{L}_1 \mathscr{D}_1 \mathscr{R}_1 \tag{2.3}$$

where $\mathscr{D}_1$ is given by

$$\mathscr{D}_1 = \begin{bmatrix} X & 0 \\ 0 & -(Y^{-1} + Z^T X^{-1} Z) \end{bmatrix}. \tag{2.4}$$

The above transformation is conducted by using X^{-1} to cancel terms. Similar approach can be applied to use Y to cancel terms, and the result would be

$$\mathscr{H} = \mathscr{L}_2 \mathscr{D}_2 \mathscr{R}_2 \tag{2.5}$$

where $\mathscr{D}_2$ is given by

$$\mathscr{D}_2 = \begin{bmatrix} X + ZYZ^T & 0 \\ 0 & -Y^{-1} \end{bmatrix}. \tag{2.6}$$

Since both $\mathscr{L}_2$ and $\mathscr{R}_2$ are invertible, the condition that X, Y and $X + ZYZ^T$ invertable indicates that $\mathscr{D}_2$ invertable. As a result of Eq. (2.5), $\mathscr{H}$ is invertable. Since $\mathscr{H}$ can also be written by Eq. (2.3) and both $\mathscr{L}_1$ and $\mathscr{R}_1$ are invertable, then $\mathscr{D}_1$ invertable, which indicates that $Y^{-1} + Z^T X^{-1} Z$ invertable.

The above elaboration proves that knowing both X and Y are invertable, then if we know one of $X + ZYZ^T$ and $Y^{-1} + Z^T X^{-1} Z$ invertable, the other one must be invertable as well. With precondition satisfied, to prove Eq. (1.1), we only need to take inverse in both Eqs. (2.3) and (2.5), and compare the upper left terms of two equations. For more details, please see **Note 5**.

Second, we elaborate the second statement. By similar approach above, we can see that the condition that X, Y and $Y + Z^T XZ$ indicates matrix $\mathscr{Q}$ invertable, where $\mathscr{Q}$ is given by

$$\mathscr{Q} = \begin{bmatrix} Y & Z^T \\ Z & -X^{-1} \end{bmatrix}. \tag{2.7}$$

Since by the first part we know that the condition that X, Y and $X + ZYZ^T$ invertable indicates that $\mathscr{H}$ invertable. To prove the second statement, we only need to find such X, Y and Z so that $\mathscr{H}$ invertable yet $\mathscr{Q}$ is not. Define $\mathscr{H}$ and $\mathscr{Q}$ as follows

$$\mathscr{H} = \begin{bmatrix} x & 2 \\ 2 & -y^{-1} \end{bmatrix}, \mathscr{Q} = \begin{bmatrix} y & 2 \\ 2 & -x^{-1} \end{bmatrix}. \tag{2.8}$$

To make $\mathscr{H}$ is invertable, we need $x \neq -4y$. To make $\mathscr{Q}$ is invertable, we need $y \neq -4x$. Those two conditions are not the same, namely, it is possible that $\mathscr{H}$ is invertable but $\mathscr{Q}$ is not. The statement is therefore proved.