KTH, SCHOOL OF ELECTRICAL ENGINEERING

On Stability Condition of Continuous Time Linear Systems

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1 PROBLEM STATEMENT

The *state space model* of a determinant continuous-time linear system is defined as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1.1}$$

It is well-known that the system is *asymptotically stable* if all the eigenvalues of A lie on the left half of complex plane and *stable* if all poles on imaginary axis have multiplicity of 1 and other eigenvalues lie on the left half of complex plane or some poles on imaginary axis have multiplicity more than 1 but A still diagonalizable.
Since the explicit solution of the state space model is given as Eq. (1.2) given by [1]

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau, \tag{1.2}$$

by **Hand Note 1**, we know the stability of the linear system only depends on a term like $e^{At}x_0$. Further elaborate the stability statement by discussing the evolvement of term $e^{At}x_0$ in the following four cases.

1. A has only real eigenvalues and is diagonalizable.
2. A has only real eigenvalues but is not diagonalizable.
3. A has some eigenvalues as complex conjugates and is diagonalizable.
4. A has some eigenvalues as complex conjugates with multiplicity more than 1, which make A not diagonalizable.

2 ELABORATION

Case 1, A has only real eigenvalues and is diagonalizable. Via diagonalization, A can be written as

$$A = PDP^{-1} \tag{2.1}$$

where $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $P = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n]$ and $P^{-1} = [\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_n]^T$. Since all eigenvalues are real, by [2] we have $P, D, P^{-1} \in M_n(\mathbb{R})$. Notice that in D it is possible that $\lambda_i = \lambda_j$ even when $i \neq j$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of P are eigenvectors of A. Therefore,

$$e^{At}x_0 = [e^{\lambda_1 t}\mathbf{v}_1\ \ e^{\lambda_2 t}\mathbf{v}_2\ \ \ldots\ \ e^{\lambda_n t}\mathbf{v}_n]P^{-1}x_0. \tag{2.2}$$

Since P^{-1} is invertible, essentially $P^{-1}x_0$ can be any vextor in $\mathbb{R}^n$. Therefore, λ_i does not have any impact on the system evolvement for all x_0 only when $\mathbf{v}_i = \mathbf{0}$, which cannot be true since P is invertible. Therefore, all eigenvalues have impact on system evolvement and it is necessary to make $\lambda_i < 0$ for $i = 1, 2, \ldots, n$. The case when $\lambda_i = 0$ would be explained in the following remark and next case.

Remark. *If 0 is an eigenvalue of a nonzore matrix A and has algebraic multiplicity as 1, the system is stable. If 0 has algebraic multiplicity more than 1, and A is diagonalizable, which means 0 has the same amount of geometrix multiplicity as its algebraic multiplicity, the system is still stable. One such example can be*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.3}$$

However, this means the system is poorly structured. Therefore, by stating 0 with multiplicity more than 1, it often means that its geometric multiplicity is 1.

Case 2 A has only real eigenvalues but is not diagonalizable. Via similarity, we have

$$A = PJP^{-1} \tag{2.4}$$

where $J = \text{diagblock}(J_1, J_2, \ldots, J_m)$, where each Jordan block J_i of J has dimension $k_i \times k_i$, and $P = [V_1\ V_2\ \ldots\ V_m]$ where V_i is composed by generalized eigenvextors corresponding to λ_i, namely

$$V_i = [\mathbf{v}_{i1}\ \mathbf{v}_{i2}\ \ldots\ \mathbf{v}_{ik_i}]. \tag{2.5}$$

Then we have

$$e^{At}x_0 = [V_1\ V_2\ \ldots\ V_m]\text{diagblock}(e^{J_1 t}, e^{J_2 t}, \ldots, e^{J_m t})P^{-1}x_0. \tag{2.6}$$

By **Hand Note 1**, we have

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \frac{t^2}{2!}e^{\lambda_i t} & \cdots & \frac{t^{k_i-1}}{(k_i-1)!}e^{\lambda_i t} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!}e^{\lambda_i t} \\ & & & \ddots & te^{\lambda_i t} \\ 0 & & & & e^{\lambda_i t} \end{bmatrix} \tag{2.7}$$

Therefore, we have

$$e^{At}x_0 = [V_1 e^{J_1 t} \;\; V_2 e^{J_2 t} \;\; \dots \;\; V_m e^{J_m t}]P^{-1}x_0. \tag{2.8}$$

where $V_i e^{J_i t}$ is a $n \times k_i$ matrix and

$$V_i e^{J_i t} = [e^{\lambda_i t}\mathbf{v}_{i1} \;\; te^{\lambda_i t}\mathbf{v}_{i1} + e^{\lambda_i t}\mathbf{v}_{i2} \;\; \dots \;\; \frac{t^{k_i-1}}{(k_i-1)!}e^{\lambda_i t}\mathbf{v}_{i1} + \cdots + e^{\lambda_i t}\mathbf{v}_{ik_i}]. \tag{2.9}$$

Essentially $P^{-1}x_0$ can be any vextor in $\mathbb{R}^n$. Therefore, λ_i does not have any impact on the system evolvement for all x_0 only when V_i is zero matrix, which cannot be true since P is invertible. Therefore, all eigenvalues have impact on system evolvement and it is necessary to make $\lambda_i < 0$ for $i = 1, 2, \dots, m$. Besides, the t^{k_i-1} would always exists for some x_0 due to that P is invertible, which means $\mathbf{v}_{i1} \neq \mathbf{0}$.

Remark. *If 0 is an eigenvalue of a nonzore matrix A and has algebraic multiplicity more than 1 and geometric multiplicity 1, then the system is unstable. To see this, take $\lambda_i = 0$ to Eq.* (2.9)*, we have*

$$V_i e^{J_i t} = [\mathbf{v}_{i1} \;\; t\mathbf{v}_{i1} + \mathbf{v}_{i2} \;\; \dots \;\; \frac{t^{k_i-1}}{(k_i-1)!}\mathbf{v}_{i1} + \cdots + \mathbf{v}_{ik_i}]. \tag{2.10}$$

which may have terms like t^{k_i-1}. Therefore it's unstable.

Case 3 A has some eigenvalues as complex conjugates and is diagonalizable. Via diagonalization, A can be written as $A = PDP^{-1}$ where $D = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, $P = [\mathbf{v}_1 \; \mathbf{v}_2 \; \dots \; \mathbf{v}_n]$ and $P^{-1} = [\mathbf{u}_1 \; \mathbf{u}_2 \; \dots \; \mathbf{u}_n]^T$. Without loss of generality, we suppose that λ_1 and λ_2 are complex conjugate and by $\mathbf{v}_1$ and $\mathbf{v}_2$ as their eigenvectors respectively. By **Note 3** we know that $\mathbf{v}_2 = \bar{\mathbf{v}}_1$. Denote $\lambda_1 = a + jb$, then $\lambda_2 = a - jb$. Then similar as Eq. (2.2), $e^{At}x_0$ can be written as

$$e^{At}x_0 = [e^{(a+jb)t}\mathbf{v}_1 \;\; e^{(a-jb)t}\bar{\mathbf{v}}_1 \;\; \dots \;\; e^{\lambda_n t}\mathbf{v}_n][\mathbf{u}_1 \; \mathbf{u}_2 \; \dots \; \mathbf{u}_n]^T x_0. \tag{2.11}$$

Denote $\mathbf{u}_i = [u_{i1} \; u_{i2}, \dots, u_{in}]^T$, then Eq. (2.11) can be written as

$$e^{At}x_0 = [\Upsilon_1 \;\; \Upsilon_2 \; \dots \; \Upsilon_n]x_0. \tag{2.12}$$

where

$$\Upsilon_i = u_{1i}\mathbf{v}_1 e^{(a+jb)t} + u_{2i}\bar{\mathbf{v}}_1 e^{(a-jb)t} + \Sigma_i. \tag{2.13}$$

In Eq. (2.13), Σ_i is the weighted sum of the rest $n-2$ eigenvectors. Since $e^{At} \in M_n(\mathbb{R})$, we have

$$\mathrm{Im}(u_{1i}\mathbf{v}_1 e^{(a+jb)t} + u_{2i}\bar{\mathbf{v}}_1 e^{(a-jb)t}) = \mathbf{0} \implies u_{2i} = \bar{u}_{1i} \implies \mathbf{u}_2 = \bar{\mathbf{u}}_1$$

Since $e^{(a\pm jb)t} = e^{at}(\cos bt \pm j \sin bt)$, then

$$\Upsilon_i = 2e^{at}\mathrm{Re}(u_{1i}\mathbf{v}_1(\cos bt + j\sin bt)) + \Sigma_i. \tag{2.14}$$

Therefore, $a \pm jb$ does not have any impact on the system evolvement for all x_0 only when $\mathbf{v}_1 = \mathbf{0}$, which cannot be true since P is invertible, or $\mathbf{u}_1 = \mathbf{0}$, which cannot be true since P^{-1} is invertible. Therefore, eigenvalues which are complex conjugate while A is diagonalizable have impact on the system and to make system asymptotically stable, we need $a < 0$.

Case 4 A has some eigenvalues as complex conjugates with multiplicity more than 1, which make A not diagonalizable. Via similarity, we have $A = PJP^{-1}$. The notations are similar as in case 2. $J = \text{diagblock}(J_1, J_2, \ldots, J_m)$ and $P = [V_1 \ V_2 \ \ldots \ V_m]$ where V_i is composed by generalized eigenvextors corresponding to λ_i. Without loss of generality, we suppose that Jordan block J_1 and J_2 have eigenvalues as conjugate pairs and therefore also the same dimension as $k \times k$. The eigenvalues are $\lambda_1 = a + jb$ and $\lambda_2 = a - jb$ respectively. $P^{-1} = [U_1 \ U_2 \ \ldots \ U_m]^T$ where $U_1 = [\mathbf{u}_{11} \ \mathbf{u}_{12} \ \ldots \ \mathbf{u}_{1k}]$ and $U_2 = [\mathbf{u}_{21} \ \mathbf{u}_{22} \ \ldots \ \mathbf{u}_{2k}]$. We know that

$$e^{At}x_0 = [V_1 e^{J_1 t} \ \ V_2 e^{J_2 t} \ \ \ldots \ \ V_m e^{J_m t}]P^{-1}x_0.$$

Besides, since λ_1 and λ_2 are complex conjugate, we can find that $V_2 = \bar{V}_1$. Furthermore,

$$V_1 e^{J_1 t} = [e^{(a+jb)t}\mathbf{v}_{11} \ \ te^{(a+jb)t}\mathbf{v}_{11} + e^{(a+jb)t}\mathbf{v}_{12} \ \ \ldots \ \ \frac{t^{k-1}}{(k-1)!}e^{(a+jb)t}\mathbf{v}_{11} + \cdots + e^{(a+jb)t}\mathbf{v}_{1k}] \tag{2.15}$$

and

$$V_2 e^{J_2 t} = [e^{(a-jb)t}\bar{\mathbf{v}}_{11} \ \ te^{(a-jb)t}\bar{\mathbf{v}}_{11} + e^{(a-jb)t}\bar{\mathbf{v}}_{12} \ \ \ldots \ \ \frac{t^{k-1}}{(k-1)!}e^{(a-jb)t}\bar{\mathbf{v}}_{11} + \cdots + e^{(a-jb)t}\bar{\mathbf{v}}_{1k}] \tag{2.16}$$

Denote $\mathbf{u}_{1i} = [\alpha_{i1} \ \alpha_{i2} \ \ldots \ \alpha_{in}]^T$ and $\mathbf{u}_{2i} = [\beta_{i1} \ \beta_{i2} \ \ldots \ \beta_{in}]^T$. Reformulate relative terms, we get

$$e^{At}x_0 = [\Upsilon_1 \ \ \Upsilon_2 \ \ldots \ \Upsilon_n]x_0$$

where

$$\begin{aligned}\Upsilon_i =& \alpha_{1i}e^{(a+jb)t}\mathbf{v}_{11} + \beta_{1i}e^{(a-jb)t}\bar{\mathbf{v}}_{11} + \alpha_{2i}e^{(a+jb)t}(t\mathbf{v}_{11} + \mathbf{v}_{12}) + \beta_{2i}e^{(a-jb)t}(t\bar{\mathbf{v}}_{11} + \bar{\mathbf{v}}_{12}) + \cdots + \\ & \alpha_{ki}e^{(a+jb)t}(\frac{t^{k-1}}{(k-1)!}\mathbf{v}_{11} + \cdots + \mathbf{v}_{1k}) + \beta_{ki}e^{(a-jb)t}(\frac{t^{k-1}}{(k-1)!}\bar{\mathbf{v}}_{11} + \cdots + \bar{\mathbf{v}}_{1k}) + \Sigma_i\end{aligned} \tag{2.17}$$

Since $e^{At} \in M_n(\mathbb{R})$, we have $\mathbf{u}_{2i} = \bar{\mathbf{u}}_{1i}$. Then

$$\begin{aligned}\Upsilon_i =& 2e^{at}\text{Re}(\alpha_{1i}e^{jbt}\mathbf{v}_{11}) + 2e^{at}\text{Re}(\alpha_{2i}e^{jbt}(t\mathbf{v}_{11} + \mathbf{v}_{12})) + \cdots + \\ & 2e^{at}\text{Re}(\alpha_{ki}e^{jbt}(\frac{t^{k-1}}{(k-1)!}\mathbf{v}_{11} + \cdots + \mathbf{v}_{1k})) + \Sigma_i\end{aligned} \tag{2.18}$$

Therefore, $a \pm jb$ does not have any impact on the system evolvement for all x_0 only when V_1 is zero matrix, which cannot be true since P is invertible, or U_1 is zero matrix, which cannot be true since P^{-1} is invertible. Therefore, eigenvalues which are complex conjugate while A is diagonalizable have impact on the system and to make system asymptotically stable, we need $a < 0$. Besides, since $\mathbf{v}_{11} \neq \mathbf{0}$, t^{k-1} always show up for some x_0.

Remark. *In this case, if $a = 0$, the system is still unstable.*

References

[1] Torkel Glad and Lennart Ljung, *Control theory: Multivariable and nonlinear methods*, CRC press, 2000.

[2] Roger A. Horn and Charles R. Johnson, *Matrix analysis*, Cambridge University Press, 2012.