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On Calculation of multivariable ODE

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1 Problem statement

The *state space model* of a determinant linear system is defined as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1.1}$$

Prove that the explicit solution of the state space model is given as Eq. (1.2) given by [1].

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau. \tag{1.2}$$

2 Elaboration

First, we give the full proof. Denote $f(t) = e^{At}$. By **Note 1**, we know the following properties:

1. $Ae^{At} = e^{At}A$. Naturally $Ae^{-At} = e^{-At}A$.
2. $e^{P^{-1}AP} = P^{-1}e^{A}P$.
3. $e^{A(s+t)} = e^{As}e^{At}$. A conclusion from it is that $(e^{-A})^{-1} = e^{A}$.
4. $e^{A+B} = e^{A}e^{B}$ if $AB = BA$, in other words, A and B commute.
5. $f'(t) = Ae^{At}$.

Then for the ODE, left multiply e^{-At} on both sides, we have

$$\dot{x}(t) - Ax(t) = Bu(t) \implies e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}Bu(t)$$

$$\implies e^{-At}\dot{x}(t) - Ae^{-At}x(t) = e^{-At}Bu(t)$$

Notice that the left hand side of the equation above is the derivative of $g(t) = e^{-At}x(t)$. Therefore we have

$$\frac{d}{dt}g(t) = e^{-At}Bu(t). \tag{2.1}$$

Taking integration from t_0 to t on both sides, we have

$$e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau \implies e^{-At}x(t) = e^{-At_0}x(t_0) + \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau.$$

Multiply e^{At} on both sides, we have Eq. (1.2).
Second, we give intuitive explanation.
(**1**) In the case where A is diagonalizable, namely $D = P^{-1}AP$, where D is a diagonal matrix, then define $x = Pz$. Use z to replace x and Eq. (1.1) can be rewritten as

$$P\dot{z}(t) = APz(t) + Bu(t) \implies \dot{z}(t) = P^{-1}APz(t) + P^{-1}Bu(t) \implies \dot{z}(t) = Dz(t) + P^{-1}Bu(t)$$

The multivariable ODE is clearly decoupled as single variable ODEs. Therefore, the formula of single ODE is applied to solve $z(t)$, which yields to

$$z(t) = e^{D(t-t_0)}z(t_0) + \int_{t_0}^{t} e^{D(t-\tau)}P^{-1}Bu(\tau)d\tau.$$

Replace z with x, we get

$$P^{-1}x(t) = e^{D(t-t_0)}P^{-1}x(t_0) + \int_{t_0}^{t} e^{D(t-\tau)}P^{-1}Bu(\tau)d\tau.$$

Multiply P on both sides, we get

$$x(t) = Pe^{D(t-t_0)}P^{-1}x(t_0) + \int_{t_0}^{t} Pe^{D(t-\tau)}P^{-1}Bu(\tau)d\tau.$$

By applying property 2 listed above, we can get Eq. (1.1).
(**2**) In the case where A is not diagonalizable, then we have $J_c = P^{-1}AP$, where J_c is Jordan canonical form. If the solution Eq. (1.2) holds for the ODE

$$\dot{z}_c(t) = J_c z_c(t) + H_c u_c(t), \tag{2.2}$$

then for ODE with any A, it can be considered as linear transformation of Eq. (2.2). By the same reasoning used in (**1**), the solution is proved. In fact, in J_c, only its Jordan block with order higher than 1 is an issue since the one dimension Jordan block still corresponds to single variable ODE, the solution of which is known. Therefore we attack the problem where the Jordan block have order higher than 1, namely, prove the solution Eq. (1.2) holds for the ODE

$$\dot{z}(t) = Jz(t) + Hu(t), \tag{2.3}$$

where $J \in M_n$ is Jordan block matrix.
By Eq. (1.2), the solution of Eq. (2.3) would be

$$z(t) = e^{J(t-t_0)}z(t_0) + \int_{t_0}^{t} e^{J(t-\tau)}Hu(\tau)d\tau. \tag{2.4}$$

Note 2 provides a great way of calculating e^J. Since $J = \lambda I + N$ where N is a nilpotent matrix. Because λI and N commute, by property 4, we know $e^{\lambda I+N} = e^{\lambda I}e^N$. Since $N^n = 0$, e^N is fairly simple to calculate. However, attention need to be paid when calculating $e^{N(t-t_0)}$, which is

$$e^{N(t-t_0)} = I + N(t-t_0) + \frac{1}{2!}N^2(t-t_0)^2 + \frac{1}{3!}N^3(t-t_0)^3 + \cdots + \frac{1}{(n-1)!}N^{n-1}(t-t_0)^{n-1}$$

where the $t - t_0$ shall not be lost. By this method, also refer to **Stochastic Dynamic Systems, UU, Note 32**, $e^{J(t-t_0)}$ would be

$$e^{J(t-t_0)} = \begin{bmatrix} e^{\lambda(t-t_0)} & (t-t_0)e^{\lambda(t-t_0)} & \frac{(t-t_0)^2}{2!}e^{\lambda(t-t_0)} & \cdots & \frac{(t-t_0)^{n-1}}{(n-1)!}e^{\lambda(t-t_0)} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{(t-t_0)^2}{2!}e^{\lambda(t-t_0)} \\ & & & \ddots & (t-t_0)e^{\lambda(t-t_0)} \\ 0 & & & & e^{\lambda(t-t_0)} \end{bmatrix} \tag{2.5}$$

The same format for $e^{J(t-\tau)}$ in Eq. (2.4). Therefore the task is to prove the solution of ODE (2.3) has solution as Eq. (2.4) where the Jordan matrix exponential in Eq. (2.4) has the format as Eq. (2.5). We denote $v(t) = Hu(t)$ and m-th item of $v(t)$ as $v_m(t)$. Then the m-th element of $z(t)$, denoted as $z_m(t)$, should be given as

$$\begin{aligned} z_m(t) =& e^{\lambda(t-t_0)}z_m(t_0) + (t-t_0)e^{\lambda(t-t_0)}z_{m+1}(t_0) + \frac{(t-t_0)^2}{2!}e^{\lambda(t-t_0)}z_{m+2}(t_0) + \cdots + \\ & \frac{(t-t_0)^{n-m}}{(n-m)!}e^{\lambda(t-t_0)}z_n(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)}v_m(\tau)d\tau + \int_{t_0}^{t}(t-\tau)e^{\lambda(t-\tau)}v_{m+1}(\tau)d\tau + \\ & \int_{t_0}^{t}\frac{(t-\tau)^2}{2!}e^{\lambda(t-\tau)}v_{m+2}(\tau)d\tau + \cdots + \int_{t_0}^{t}\frac{(t-\tau)^{n-m}}{(n-m)!}e^{\lambda(t-\tau)}v_n(\tau)d\tau. \end{aligned} \tag{2.6}$$

We prove it by induction and start with the last element of $z(t)$, denoted as $z_n(t)$.

1. For $z_n(t)$, the ODE is $\dot{z}_n(t) = \lambda z_n(t) + v_n(t)$. Then the solution would be

$$z_n(t) = e^{\lambda(t-t_0)}z_n(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)}v_n(\tau)d\tau. \tag{2.7}$$

   which fulfill Eq. (2.6).

2. For $z_{n-1}(t)$, the ODE is $\dot{z}_{n-1}(t) = \lambda z_{n-1}(t) + z_n(t) + v_{n-1}(t)$. Consider $z_n(t) + v_{n-1}(t) = w_{n-1}(t)$, then ODE here is the same as in step 1. Therefore, we can still use single variable ODE solution to solve $z_{n-1}(t)$, which gives

$$z_{n-1}(t) = e^{\lambda(t-t_0)} z_{n-1}(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)} (z_n(\tau) + v_{n-1}(\tau)) d\tau. \tag{2.8}$$

Take Eq. (2.7) into Eq. (2.8), we get

$$\begin{aligned} z_{n-1}(t) =& e^{\lambda(t-t_0)} z_{n-1}(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)} v_{n-1}(\tau) d\tau + \int_{t_0}^{t} e^{\lambda(t-\tau)} z_n(\tau) d\tau \\ =& e^{\lambda(t-t_0)} z_{n-1}(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)} v_{n-1}(\tau) d\tau + \\ & \int_{t_0}^{t} e^{\lambda(t-\tau)} (e^{\lambda(\tau-t_0)} z_n(t_0) + \int_{t_0}^{\tau} e^{\lambda(\tau-s)} v_n(s) ds) d\tau \\ =& e^{\lambda(t-t_0)} z_{n-1}(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)} v_{n-1}(\tau) d\tau + \int_{t_0}^{t} e^{\lambda(t-\tau)} e^{\lambda(\tau-t_0)} z_n(t_0) d\tau + \\ & \int_{t_0}^{t} e^{\lambda(t-\tau)} \int_{t_0}^{\tau} e^{\lambda(\tau-s)} v_n(s) ds d\tau \\ =& e^{\lambda(t-t_0)} z_{n-1}(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)} v_{n-1}(\tau) d\tau + \int_{t_0}^{t} e^{\lambda(t-t_0)} z_n(t_0) d\tau + \\ & \int_{t_0}^{t} \int_{t_0}^{\tau} e^{\lambda(t-s)} v_n(s) ds d\tau \end{aligned} \tag{2.9}$$

The third term can be calculated as

$$\int_{t_0}^{t} e^{\lambda(t-t_0)} z_n(t_0) d\tau = e^{\lambda(t-t_0)} z_n(t_0) \int_{t_0}^{t} d\tau = (t-t_0) e^{\lambda(t-t_0)} z_n(t_0). \tag{2.10}$$

As for the last term, changing the integration sequence yields to

$$\int_{t_0}^{t} \int_{t_0}^{\tau} e^{\lambda(t-s)} v_n(s) ds d\tau = \int_{t_0}^{t} \int_{s}^{t} e^{\lambda(t-s)} v_n(s) d\tau ds = \int_{t_0}^{t} (t-s) e^{\lambda(t-s)} v_n(s) ds \tag{2.11}$$

In $\int_{t_0}^{t} (t-s) e^{\lambda(t-s)} v_n(s) ds$, changing s to τ does not alter the result, therefore we have

$$\int_{t_0}^{t} \int_{t_0}^{\tau} e^{\lambda(t-s)} v_n(s) ds d\tau = \int_{t_0}^{t} (t-\tau) e^{\lambda(t-\tau)} v_n(\tau) d\tau \tag{2.12}$$

Take the results of Eq. (2.10) and Eq. (2.12) into Eq. (2.9), we get

$$\begin{aligned} z_{n-1}(t) =& e^{\lambda(t-t_0)} z_{n-1}(t_0) + (t-t_0) e^{\lambda(t-t_0)} z_n(t_0) + \\ & \int_{t_0}^{t} e^{\lambda(t-\tau)} v_{n-1}(\tau) d\tau + \int_{t_0}^{t} (t-\tau) e^{\lambda(t-\tau)} v_n(\tau) d\tau \end{aligned} \tag{2.13}$$

which fulfill Eq. (2.6).

3. Suppose m+1 th term fulfill Eq. (2.6). Namely

$$\begin{aligned} z_{m+1}(t) =& e^{\lambda(t-t_0)} z_{m+1}(t_0) + (t-t_0)e^{\lambda(t-t_0)} z_{m+2}(t_0) + \frac{(t-t_0)^2}{2!} e^{\lambda(t-t_0)} z_{m+3}(t_0) + \cdots + \\ & \frac{(t-t_0)^{n-m-1}}{(n-m-1)!} e^{\lambda(t-t_0)} z_n(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)} v_{m+1}(\tau) d\tau + \\ & \int_{t_0}^{t} (t-\tau) e^{\lambda(t-\tau)} v_{m+2}(\tau) d\tau + \int_{t_0}^{t} \frac{(t-\tau)^2}{2!} e^{\lambda(t-\tau)} v_{m+3}(\tau) d\tau + \cdots + \\ & \int_{t_0}^{t} \frac{(t-\tau)^{n-m-1}}{(n-m-1)!} e^{\lambda(t-\tau)} v_n(\tau) d\tau. \end{aligned} \tag{2.14}$$

where $m < n-1$. For $z_m(t)$, the ODE is $\dot{z}_m(t) = \lambda z_m(t) + z_{m+1}(t) + v_m(t)$. Consider $z_{m+1}(t) + v_m(t) = w_m(t)$, then ODE here is the same as in step 1. Therefore, we can still use single variable ODE solution to solve $z_m(t)$, which gives

$$\begin{aligned} z_m(t) &= e^{\lambda(t-t_0)} z_m(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)} (z_{m+1}(\tau) + v_m(\tau)) d\tau \\ &= e^{\lambda(t-t_0)} z_m(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)} v_m(\tau) d\tau + \int_{t_0}^{t} e^{\lambda(t-\tau)} z_{m+1}(\tau) d\tau. \end{aligned} \tag{2.15}$$

Therefore the only term we need to deal with is

$$\int_{t_0}^{t} e^{\lambda(t-\tau)} z_{m+1}(\tau) d\tau. \tag{2.16}$$

Taking Eq. (2.14) into Eq. (2.16) is the next step. Define $f(t)$ as follow

$$\begin{aligned} f(t) =& e^{\lambda(t-t_0)} z_{m+1}(t_0) + (t-t_0) e^{\lambda(t-t_0)} z_{m+2}(t_0) + \\ & \frac{(t-t_0)^2}{2!} e^{\lambda(t-t_0)} z_{m+3}(t_0) + \cdots + \frac{(t-t_0)^{n-m-1}}{(n-m-1)!} e^{\lambda(t-t_0)} z_n(t_0). \end{aligned} \tag{2.17}$$

Define $g(t)$ as follows

$$\begin{aligned} g(t) =& \int_{t_0}^{t} e^{\lambda(t-\tau)} v_{m+1}(\tau) d\tau + \int_{t_0}^{t} (t-\tau) e^{\lambda(t-\tau)} v_{m+2}(\tau) d\tau + \\ & \int_{t_0}^{t} \frac{(t-\tau)^2}{2!} e^{\lambda(t-\tau)} v_{m+3}(\tau) d\tau + \cdots + \int_{t_0}^{t} \frac{(t-\tau)^{n-m-1}}{(n-m-1)!} e^{\lambda(t-\tau)} v_n(\tau) d\tau. \end{aligned} \tag{2.18}$$

Apparently $z_{m+1}(t) = f(t) + g(t)$ and both $f(t)$ and $g(t)$ have $n-m$ items. Denote i-th item of $f(t)$ as $f_i(t)$, namely

$$f_i(t) = \frac{(t-t_0)^{i-1}}{(i-1)!} e^{\lambda(t-t_0)} z_{m+i}(t_0) \tag{2.19}$$

Then we have

$$f(t) = \sum_{i=1}^{n-m} f_i(t) \tag{2.20}$$

Denote i-th item of $g(t)$ as $g_i(t)$, namely

$$g_i(t) = \int_{t_0}^{t} \frac{(t-\tau)^{i-1}}{(i-1)!} e^{\lambda(t-\tau)} v_{m+i}(\tau) d\tau \tag{2.21}$$

Then we have

$$g(t) = \sum_{i=1}^{n-m} g_i(t) \tag{2.22}$$

Therefore, we have

$$z_{m+1} = \sum_{i=1}^{n-m} f_i(t) + \sum_{i=1}^{n-m} g_i(t) \tag{2.23}$$

Taking Eq. (2.23) into Eq. (2.16) yields to

$$\begin{aligned} &\int_{t_0}^{t} e^{\lambda(t-\tau)} \sum_{i=1}^{n-m} f_i(\tau) d\tau + \int_{t_0}^{t} e^{\lambda(t-\tau)} \sum_{i=1}^{n-m} g_i(\tau) d\tau = \\ &\sum_{i=1}^{n-m} \int_{t_0}^{t} e^{\lambda(t-\tau)} f_i(\tau) d\tau + \sum_{i=1}^{n-m} \int_{t_0}^{t} e^{\lambda(t-\tau)} g_i(\tau) d\tau \end{aligned} \tag{2.24}$$

Therefore, we only need to prove that the conclusion holds for i-th item, namely calculating $\int_{t_0}^{t} e^{\lambda(t-\tau)} f_i(\tau) d\tau$ and $\int_{t_0}^{t} e^{\lambda(t-\tau)} g_i(\tau) d\tau$.

$$\begin{aligned} \int_{t_0}^{t} e^{\lambda(t-\tau)} f_i(\tau) d\tau &= \int_{t_0}^{t} e^{\lambda(t-\tau)} \frac{(\tau-t_0)^{i-1}}{(i-1)!} e^{\lambda(\tau-t_0)} z_{m+i}(t_0) d\tau \\ &= \int_{t_0}^{t} e^{\lambda(t-t_0)} z_{m+i}(t_0) \frac{(\tau-t_0)^{i-1}}{(i-1)!} d\tau \\ &= e^{\lambda(t-t_0)} z_{m+i}(t_0) \int_{t_0}^{t} \frac{(\tau-t_0)^{i-1}}{(i-1)!} d\tau \\ &= \frac{(t-t_0)^{i}}{i!} e^{\lambda(t-t_0)} z_{m+i}(t_0) \end{aligned} \tag{2.25}$$

which fulfill the format of Eq. (2.6). As for $\int_{t_0}^{t} e^{\lambda(t-\tau)} g_i(\tau) d\tau$, we have

$$\begin{aligned} \int_{t_0}^{t} e^{\lambda(t-\tau)} g_i(\tau) d\tau &= \int_{t_0}^{t} e^{\lambda(t-\tau)} \int_{t_0}^{\tau} \frac{(\tau-s)^{i-1}}{(i-1)!} e^{\lambda(\tau-s)} v_{m+i}(s) ds d\tau \\ &= \int_{t_0}^{t} \int_{t_0}^{\tau} \frac{(\tau-s)^{i-1}}{(i-1)!} e^{\lambda(t-s)} v_{m+i}(s) ds d\tau \\ &= \int_{t_0}^{t} e^{\lambda(t-s)} v_{m+i}(s) \int_{s}^{t} \frac{(\tau-s)^{i-1}}{(i-1)!} d\tau ds \\ &= \int_{t_0}^{t} \frac{(t-s)^{i}}{i!} e^{\lambda(t-s)} v_{m+i}(s) ds \\ &= \int_{t_0}^{t} \frac{(t-\tau)^{i}}{i!} e^{\lambda(t-\tau)} v_{m+i}(\tau) d\tau \end{aligned} \tag{2.26}$$

which fulfill the format given by Eq. (2.6). Therefore, the proof is concluded.

References

[1] Torkel Glad and Lennart Ljung, *Control theory: Multivariable and nonlinear methods*, CRC press, 2000.