KTH, School of Electrical Engineering

On Calculation of Jordan Block Power

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1 Problem statement

Denote $J(\lambda) \in M_n$ as a Jordan block and $N \in M_n$ the nilpotent matrix whose superdiagonal contains ones and all other entries are zero. Therefore, we have

$$J(\lambda) = \lambda I + N. \tag{1.1}$$

Exercise 7.2.19 in **Note 35** provides the results of Jordan block power $J(\lambda)^k$. Use Eq. (1.1) to prove that.

2 Elaboration

First, we make a proposition.

Proposition. *For $A, B \in M_n$, if A and B commute, namely $AB = BA$, then*

$$(A+B)^k = C_k^0 A^k + C_k^1 A^{k-1} B + \cdots + C_k^{k-1} AB^{k-1} + C_k^k B^k \tag{2.1}$$

where

$$C_k^j = \frac{k!}{j!(k-j)!},\ 0 \le j \le k. \tag{2.2}$$

Second, for I and N, apparently $IN = NI$. Therefore, according to the proposition, we have

$$J(\lambda)^k = C_k^0 (\lambda I)^k + C_k^1 (\lambda I)^{k-1} N + \cdots + C_k^{k-1} (\lambda I) N^{k-1} + C_k^k N^k. \tag{2.3}$$

Since $(\lambda I)^j = \lambda^j I$ for any $j \in \mathbb{N}$ and $N^j = 0$ for $j \ge n$. Here we only elaborate the case when $k > n$, then Eq. (2.3) can be simplified as

$$J(\lambda)^k = C_k^0 \lambda^k I + C_k^1 \lambda^{k-1} N + \cdots + C_k^{n-1} \lambda^{k-(n-1)} N^{n-1} \tag{2.4}$$

which conclude the proof.