KTH, School of Electrical Engineering

Spectrum of Output in State Space Model

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1 Problem statement

The *state space model* of a system is defined as

$$\begin{aligned} x(t+1) &= Fx(t) + v(t), \\ y(t) &= Hx(t) + e(t). \end{aligned} \tag{1.1}$$

where $v(t)$ and $e(t)$ are complex-valued uncorrelated sequences with zero mean and

$$\mathbf{E}\begin{pmatrix} v(t) \\ e(t) \end{pmatrix} (v^*(s)\ e^*(s)) = \begin{pmatrix} R_1 & R_{12} \\ R_{21} & R_2 \end{pmatrix} \delta_{t,s}. \tag{1.2}$$

According to [1], after reaching stationary state, the mean of $x(t)$ and $y(t)$ are both 0, namely $m_x = 0$ and $m_y = 0$. The covariance of $x(t)$, namely $R_x(t, t)$, denoted as P, is given as

$$P = \sum_{j=0}^{\infty} F^j R_1 F^{*j}. \tag{1.3}$$

By Eqs. (4.20) and (4.21) in [1], after reaching stationary state the output covariance function is given as

$$R_y(t, t) = HPH^* + R_2, \tag{1.4}$$

$$R_y(t+\tau, t) = HF^\tau PH^* + HF^{\tau-1}R_{12},\ \tau > 0 \tag{1.5}$$

$$R_y(t-\tau, t) = HPF^{*\tau}H^* + R_{21}F^{*(\tau-1)}H^*,\ \tau > 0 \tag{1.6}$$

The spectrum of a time series, say $\{y(t)\}$, is defined as

$$\phi_y(z) = \sum_{n=-\infty}^{\infty} R_y(t+n, t) z^{-n}. \tag{1.7}$$

By [1], in the stationary case, the spectrum of $y(t)$ is given as

$$\begin{aligned}\phi_y(z) =& H(zI-F)^{-1}R_1(z^{-*}I-F)^{-*}H^* + R_2\\ & H(zI-F)^{-1}R_{12} + R_{21}(z^{-*}I-F)^{-*}H^*,\end{aligned} \tag{1.8}$$

which is calculated via the *transfer function model* rather than state space model. Elaborate that the same result can also be given via the state space model.

2 Elaboration

Take Eqs. (1.4), (1.5) and (1.6) into Eq. (1.7), we obtain

$$\phi_y(z) = HPH^* + R_2 + \sum_{n=1}^{\infty}(HF^nPH^* + HF^{n-1}R_{12})z^{-n} + \sum_{m=1}^{\infty}(HPF^{*m}H^* + R_{21}F^{*(m-1)}H^*)z^m, \tag{2.1}$$

which can be rewritten as

$$\begin{aligned}\phi_y(z) =& HPH^* + R_2 + H(\sum_{n=1}^{\infty}F^nz^{-n})PH^* + H(\sum_{n=1}^{\infty}F^{n-1}z^{-n})R_{12}\\ &+ HP(\sum_{m=1}^{\infty}F^{*m}z^m)H^* + R_{21}(\sum_{m=1}^{\infty}F^{*(m-1)}z^m)H^*.\end{aligned} \tag{2.2}$$

First of all, we take care of $\sum F^n z^{-n}$.

$$S_n = \sum_{n=1}^{\infty}F^nz^{-n} \implies S_n = \sum_{n=1}^{\infty}(Fz^{-1})^n \implies (I-Fz^{-1})S_n = Fz^{-1} \implies (zI-F)S_n = F$$

Therefore,

$$\sum_{n=1}^{\infty}F^nz^{-n} = (zI-F)^{-1}F \tag{2.3}$$

$$\sum_{n=1}^{\infty}F^{(n-1)}z^{-n} = (\sum_{n=1}^{\infty}F^nz^{-n})F^{-1} = (zI-F)^{-1} \tag{2.4}$$

Similarly,

$$\sum_{n=0}^{\infty}F^nz^{-n} = (zI-F)^{-1}z. \tag{2.5}$$

Then for $\sum F^{*m}z^m$, we calculate $\sum F^m z^{*m}$ instead in order to follow the notation used in [1]

$$S_m^* = \sum_{m=1}^{\infty}F^mz^{*m} \implies S_m^* = \sum_{m=1}^{\infty}(Fz^*)^m \implies (I-Fz^*)S_m^* = Fz^* \implies (z^{-*}I-F)S_m^* = F$$

$$\implies S_m^* = (z^{-*}I-F)^{-1}F$$

Therefore,

$$\sum_{m=1}^{\infty}F^{*m}z^m = F^*(z^{-*}I-F)^{-*} \tag{2.6}$$

$$\sum_{m=1}^{\infty} F^{*(m-1)} z^m = F^{-*}(\sum_{m=1}^{\infty} F^{*m} z^m) = (z^{-*} I - F)^{-*} \tag{2.7}$$

Similarly,

$$\sum_{m=0}^{\infty} F^{*m} z^m = (z^{-*} I - F)^{-*} z^{-1} \tag{2.8}$$

Remark. *Since $(zI - F)^{-1}$ is the sum of Fz, so $(zI - F)^{-1}$ and F can commute, namely $(zI - F)^{-1}F = F(zI - F)^{-1}$. So do $(z^{-*} I - F)^{-*}$ and F^*. Besides, by the different approaches of calculating $\sum F^{*m} z^m$, we know $(z^{-*} I - F)^{-*} = (z^{-1} I - F^*)^{-1}$.*

Take Eqs. (2.4) and (2.7) into Eq. (2.2), we get

$$\begin{aligned} \phi_y(z) = & H(\sum_{n=1}^{\infty} F^n z^{-n} P + P + P \sum_{m=1}^{\infty} F^{*m} z^m) H^* + R_2 \\ & + H(zI - F)^{-1} R_{12} + R_{21} (z^{-*} I - F)^{-*} H^*, \end{aligned} \tag{2.9}$$

which is one step closer to Eq. (1.8). Now only the first three terms in Eq. (2.9) need to be handled. Denote HTH^* as the sum of those terms, namely

$$T = \sum_{n=1}^{\infty} F^n z^{-n} P + P + P \sum_{m=1}^{\infty} F^{*m} z^m. \tag{2.10}$$

Take Eq. (1.3) into Eq. (2.10) and change the order of sum operator, we get

$$T = \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} F^n z^{-n} F^j R_1 F^{*j} + \sum_{j=0}^{\infty} F^j R_1 F^{*j} + \sum_{j=0}^{\infty} \sum_{m=1}^{\infty} F^j R_1 F^{*j} F^{*m} z^m. \tag{2.11}$$

Denote the first term in Eq. (2.11) T_f and the sum of the rest two terms as T_b. So $T = T_f + T_b$. Then we have

$$T_f = \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} F^n z^{-n} F^j z^{-j} R_1 F^{*j} z^j = \sum_{j=0}^{\infty} (\sum_{n=j+1}^{\infty} (F z^{-1})^n) R_1 (F^* z)^j. \tag{2.12}$$

For T_b, we change the order of sum operator to arrange it as

$$\begin{aligned} T_b &= \sum_{j=0}^{\infty} (F z^{-1})^j R_1 (F^* z)^j + \sum_{j=0}^{\infty} ((F z^{-1})^j R_1 \sum_{m=j+1}^{\infty} (F^* z)^m) \\ &= \sum_{j=0}^{\infty} (F z^{-1})^j R_1 (F^* z)^j + \sum_{m=1}^{\infty} (\sum_{j=0}^{m-1} (F z^{-1})^j) R_1 (F^* z)^m. \end{aligned} \tag{2.13}$$

We change the subscripts in the first sum of Eq. (2.13) from j to m and split the case when $m = 0$, then

$$\begin{aligned} T_b &= (F z^{-1})^0 R_1 (F^* z)^0 + \sum_{m=1}^{\infty} (F z^{-1})^m R_1 (F^* z)^m + \sum_{m=1}^{\infty} (\sum_{j=0}^{m-1} (F z^{-1})^j) R_1 (F^* z)^m \\ &= (F z^{-1})^0 R_1 (F^* z)^0 + \sum_{m=1}^{\infty} (\sum_{j=0}^{m} (F z^{-1})^j) R_1 (F^* z)^m \\ &= \sum_{m=0}^{\infty} (\sum_{j=0}^{m} (F z^{-1})^j) R_1 (F^* z)^m \end{aligned} \tag{2.14}$$

Again we change the subscripts in the sums of Eqs. (2.12) and (2.14). Use k for the outer sum and l for the inner sum, then

$$
\begin{aligned}
T &= \sum_{k=0}^{\infty}(\sum_{l=k+1}^{\infty}(Fz^{-1})^{l})R_1(F^*z)^k + \sum_{k=0}^{\infty}(\sum_{l=0}^{k}(Fz^{-1})^{l})R_1(F^*z)^k \\
&= \sum_{k=0}^{\infty}(\sum_{l=0}^{\infty}(Fz^{-1})^{l})R_1(F^*z)^k \\
&= \sum_{l=0}^{\infty}(Fz^{-1})^{l}R_1\sum_{k=0}^{\infty}(F^*z)^k. \qquad (2.15)
\end{aligned}
$$

Take Eqs. (2.5) and (2.8) into Eq. (2.15), we have

$$
\begin{aligned}
T &= (zI-F)^{-1}zR_1(z^{-*}I-F)^{-*}z^{-1} \\
&= (zI-F)^{-1}R_1(z^{-*}I-F)^{-*} \qquad (2.16)
\end{aligned}
$$

which conclude the proof.

References

[1] Torsten Söderström, *Discrete-time stochastic systems: Estimation and control*, Springer & Verlag, 2002.