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On the Solution of Sylvester's Equation

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August 10, 2017

1 Problem statement

A *Sylvester's equation* is defined as

$$AX - XB = C \tag{1.1}$$

where $A \in M_n$, $B \in M_m$, $X \in M_{n,m}$ and $C \in M_{n,m}$. It is claimed in [1], **Theroem 2.4.4.1** that for any given $C \in M_{n,m}$, there exists an unique $X \in M_{n,m}$ as the solution of Eq. (1.1) if and only if $\sigma(A) \cap \sigma(B) = \emptyset$.

The proof directly follows the theorem statement in [1]. The authors claim that for the theorem to hold, it suffices to show that the only solution of $AX - XB = 0$ is 0. This statement is obviously true for the uniqueness part, that is after showing that the only solution of $AX - XB = 0$ is 0, if $AX - XB = C$ has solution, then it must be unique. However, it is less obvious why the statement also elaborate the existence of X to Eq. (1.1). Elaborate this aspect. Prove that the proof in [1] also takes care of the existence of solution.

2 Elaboration

By straightforward calculation, it is easy to show that solving Eq. (1.1) is equivalent to solving simultaneous equations

$$a_{i1}x_{1j} + a_{i2}x_{2j} + \cdots + a_{in}x_{nj} - (x_{i1}b_{1j} + x_{i2}b_{2j} + \cdots + x_{im}b_{mj}) = c_{ij} \tag{2.1}$$

for all $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m$, where a_{ij}, b_{ij}, c_{ij}, x_{ij} are the element of A, B, C, X respectively.

Denote

$$X = [\mathbf{x}_1\, \mathbf{x}_2 \cdots \mathbf{x}_m],\ \mathbf{x}_j \in \mathbb{C}^n,\ j = 1,2,\ldots,m;$$

$$C = [\mathbf{c}_1\, \mathbf{c}_2 \cdots \mathbf{c}_m],\ \mathbf{c}_j \in \mathbb{C}^n,\ j = 1, 2, \ldots, m.$$

Define $\mathbf{v}^T = [\mathbf{x}_1^T\, \mathbf{x}_2^T \cdots \mathbf{x}_m^T]$ and $\mathbf{y}^T = [\mathbf{c}_1^T\, \mathbf{c}_2^T \cdots \mathbf{c}_m^T]$. Apparantly, there exists an unique matrix $D \in M_{nm,nm}$, such that solving

$$D\mathbf{v} = \mathbf{y} \tag{2.2}$$

is equivalent to solving simultaneous equations (2.1). Therefore, solving the Sylvester's equation, namely Eq. (1.1) is equivalent to solving Eq. (2.2). To prove Eq. (2.2) has an unique solution $\mathbf{v} \in \mathbb{C}^{nm}$ for any $\mathbf{y} \in \mathbb{C}^{nm}$, it suffices to show that the only solution to $D\mathbf{v} = 0$ is $\mathbf{v} = 0$, which also ensures the existence of solution for equation $D\mathbf{v} = \mathbf{y}$, refer to [1] **0.5 Nonsingularity** and [2]. Therefore, to prove Sylvester's equation has an unique solution, it is also suffice to show that the only solution of $AX - XB = 0$ is 0.

References

[1] Roger A. Horn and Charles R. Johnson, *Matrix analysis*, Cambridge University Press, 2012.

[2] Baodong Zheng, *Linear algebra and analytical geometry (Chinese)*, Higher Education Press, 2008.