KTH, School of Electrical Engineering

Proof of Parseval's Theorem

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1 Problem statement

Suppose $A(x)$ and $B(x)$ are two square integrable (with respect to the Lebesgue measure), complex-valued functions on $\mathbb{C}$ of period 2π with Fourier series

$$A(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} \tag{1.1}$$

and

$$B(x) = \sum_{n=-\infty}^{\infty} b_n e^{inx} \tag{1.2}$$

respectively. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} A(x)\bar{B}(x)\mathrm{d}x = \sum_{n=-\infty}^{\infty} a_n \bar{b}_n. \tag{1.3}$$

Prove the theorem.

2 Elaboration

The result is a natural result of *mutual orthogonality* of sine and cosine functions. Since

$$\int_{-\pi}^{\pi} e^{inx}e^{-imx}\mathrm{d}x = \int_{-\pi}^{\pi} e^{i(n-m)x}\mathrm{d}x = \int_{-\pi}^{\pi} \cos((n-m)x) + i\sin((n-m)x)\mathrm{d}x. \tag{2.1}$$

The result is 2π when $n-m=0$ and 0 otherwise. Therefore, apply Eqs. (1.1) and (1.2) into the left hand side of Eq. (1.3), we have

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} A(x)\bar{B}(x)\mathrm{d}x &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n\bar{b}_m e^{i(n-m)x}\mathrm{d}x \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi} a_n\bar{b}_m e^{i(n-m)x}\mathrm{d}x.
\end{aligned}
$$

Sine only when $n = m$ the term would preserve so the proof is done.