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On Linear Estimator Formula

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1 PROBLEM STATEMENT

Suppose X, Y_1, $Y_2,\ldots,$ Y_n are random scalars and Y_1, $Y_2,\ldots,$ Y_n have finite second moments. According to [1], the linear estimation of X based on Y_1, $Y_2,\ldots,$ Y_n is defined as

$$\widehat{E}[X|Y] = E[X] + \mathrm{Cov}[X,Y]\mathrm{Cov}[Y]^{-1}(Y - E[Y]). \tag{1.1}$$

This requires that Cov[Y] is not singular, which means $\mathrm{Cov}[Y]^{-1}$ exists. Prove that Cov[Y] is not singular only if none of Y_i can be written as linear combination of the other random scalars.

2 ELABORATION

Without loss of generality, we suppose that

$$Y_1 = kZ. \tag{2.1}$$

where $Z^T = [Y_2, Y_3,\ldots, Y_n]$ and Cov[Z] is not singular. Then Cov[Y] is calculated as

$$\mathrm{Cov}[Y] = \begin{bmatrix} \mathrm{Cov}[Y_1] & \mathrm{Cov}[Y_1,Z] \\ \mathrm{Cov}[Z,Y_1] & \mathrm{Cov}[Z] \end{bmatrix} = \begin{bmatrix} k\mathrm{Cov}[Z]k^T & k\mathrm{Cov}[Z] \\ \mathrm{Cov}[Z]k^T & \mathrm{Cov}[Z] \end{bmatrix}$$

Since Cov[Z] is not singular, according to **Note 11**, the *Schur Diagonalization* of Cov[Y] is calculated as

$$\mathrm{Cov}[Y] = \begin{bmatrix} k\mathrm{Cov}[Z]k^T & k\mathrm{Cov}[Z] \\ \mathrm{Cov}[Z]k^T & \mathrm{Cov}[Z] \end{bmatrix} = \begin{bmatrix} I & k \\ 0 & I \end{bmatrix} \begin{bmatrix} M/\mathrm{Cov}[Z] & 0 \\ 0 & \mathrm{Cov}[Z] \end{bmatrix} \begin{bmatrix} I & 0 \\ k^T & I \end{bmatrix}$$

Since

$$M/\mathrm{Cov}[Z] = k\mathrm{Cov}[Z]k^T - k\mathrm{Cov}[Z]\mathrm{Cov}[Z]^{-1}\mathrm{Cov}[Z]k^T = 0,$$

so

$$\mathrm{Cov}[Y] = \begin{bmatrix} I & k \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Cov}[Z] \end{bmatrix} \begin{bmatrix} I & 0 \\ k^T & I \end{bmatrix}$$

is singular in this case.
Therefore the linear estimator formula indicates that none of Y_i can be linear defined by the rest.

References

[1] Bruce Hajek, *Random processes for engineers*, Cambridge University Press, 2015.