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On Absolute and Square Summability

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1 Problem statement

An MA(∞) is given by Eq. (1.1), where L is Lag operator and $\epsilon_t \sim \mathbf{N}(0, \delta^2)$ is *i.i.d.*

$$y_t = \sum_{i=0}^{\infty} \phi_i L^i \epsilon_{t-i} \tag{1.1}$$

By **Note 19**, the *convergence* of series $\{y_t\}$ is a precondition for any further discussion regarding stationary process. It turns out that the very same condition also ensures the existence of stationary solution to the difference equation Eq. (1.1). The condition is illustrated as in Eq. (1.2), which means the infinite series $\{\phi_i\}$ has *square summability.*

$$\sum_{i=0}^{\infty} \phi_i^2 < \infty \tag{1.2}$$

The proof of the condition can be found in **Note 19** (similar consideration regarding convergence is given in **Note 8** when $(1-\phi L)^{-1}$ is defined), and the convergence of *Cauchy sequences* are used in both **Note 19** and **Note 8**. Refer to **Note 9** for proof of the convergence of Cauchy sequences.
Absolute summability of infinite series $\{\phi_i\}$ is defined as Eq. (1.3).

$$\sum_{i=0}^{\infty} |\phi_i| < \infty \tag{1.3}$$

It is also stated in **Note 19** that absolute summability is a slight stronger condition than square summability.
Prove that absolute summability is stronger than square summability by elaborating the following two aspects.

1. Any absolute-summable series is square-summable.

2. There exists a series which is square-summable but not absolute-summable.

2 Elaboration

First, the proof of the proposition that any absolute-summable series is certainly square-summable can be found in **Note 21**.
Second, the series which is not absolute-summable but square summable is denoted as $\{d_i\}$, defined as

$$d_i = \frac{1}{i+1},\ i = 0, 1, \cdots$$

whose (absolute) sum is *Harmonic series*, given as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

The Harmonic series is divergent, so as the absolute sum of $\{d_i\}$. The proof of Harmonic series divergence can be found in **Note 22**. The square sum of $\{d_i\}$, which is called *Basel series*, given as

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

is convergent. Solving the convergent limit of the Basel series is called the *Basel problem*. The proof is given in **Note 23**, in which the proof via Fourier series needs further attention. It applies the *Parseval's identity* for Fourier series, whose proof can be found in **Note 24**. The proof uses *Mutual orthogonality* of sine and cosine functions, and the feature is elaborated in **Note 25**.