KTH, SCHOOL OF ELECTRICAL ENGINEERING

On Hermitian Matrix and Positive-definite Feature

Yuchao Li

July 20, 2017

1 PROBLEM STATEMENT

By definition, an $n \times n$ *Hermitian Matrix* H is *positive definite* if for any non-zero vector $z \in \mathbb{C}^n$, the scalar $z^* Hz$ is real and positive. This implies that any positive definite matrix is Hermitian, which is not obvious. This discussion shows the following two statement are equivalent:

a) H is a Hermitian matrix with positive eigenvalues.

b) H is a positive definite matrix.

2 ELABORATION

First, we prove that a) ⇒ b). Since H is Hermitian, by **Note 2**, H can be diagonalized as a diagonal matrix D by an unitary matrix U, namely $H = U^* DU$, where

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Since the eigenvalues of H are positive, $S = \sqrt{D}$ exists and S is Hermitian due to that $S = S^*$. Therefore, for any non-zero vector $z \in \mathbb{C}^n$,

$$z^* Hz = z^* U^* S^* SUz = (SUz)^* SUz > 0$$

Therefore, H is an positive definite matrix.
Second, we prove that b) $\Rightarrow$ a). Step one is to prove positive definite matrix H must be Hermitian. Since $z^* Hz$ is a real scalar for all $z \in \mathbb{C}^n$, therefore, $z^* Hz = (z^* Hz)^*$. Since $(z^* Hz)^* = z^* H^* z$, therefore, $z^* Hz - z^* H^* z = 0$, which leads to that $z^*(H - H^*)z = 0$. Therefore $H = H^*$. An alternative interpretation can be that since $z^* Hz$ is a real scalar, $z_i^* h_{ij} z_j$ and $z_j^* h_{ji} z_i$ need to be conjugate in order to cancel the imaginary part. This leads to that H being Hermitian. Step two is to prove that H has positive eigenvalues. Since $H = U^* DU$, $z^* Hz = z^* U^* DUz$. Denote $k = Uz$, then

$$z^* Hz = k^* Dk = \sum_{i=1}^{n} \lambda_i |k_i|^2$$

where k_i is the element of vector k. Since $z^* Hz > 0$, $\lambda_i > 0$ for $i = 1, \cdots n$. Namely, H has positive eigenvalues.

Remark. *A positive definite matrix H can always be written as $A^* A$ where $A = SU$.*