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On the Autoregressive Feature of ARMA

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1 PROBLEM STATEMENT

The discussion here uses an AR(1), given by Eq. (1.1), to discuss the *Autoregressive* feature of ARMA process, where $\epsilon_t \sim \mathbf{N}(0, \delta^2)$ is *i.i.d.* *Stationary process* in the following discussion refers to *weakly* stationary process. The discussion here is limited to the scale case.

$$y_t = \phi y_{t-1} + \epsilon_t \tag{1.1}$$

Elaborate the following two aspects.

1. The condition for the existence of the *stationary point* in Eq. (1.1) and uniqueness of such a point. The stationary point here refers to a stationary process which is the solution of the difference equation Eq. (1.1).

2. The autoregressive feature of AR(1). Namely, y_t will converge to the same distribution regardless of the starting point y_0 given the condition that the stationary point exists. We name the distribution that y_t converges to as *convergent point*.

2 ELABORATION

First, we discuss the existence of the stationary point. Assume the time series y_t is stationary and $\mathbf{E}y_t = \mu$. Take expectation on both sides of Eq. (1.1), we obtain $\mu = \phi\mu$. This indicate that the mean of stationary y_t must be 0, namely $\mu = 0$. The variance of y_t is given by

$$\gamma(0) = \mathbf{E}y_t^2 = \mathbf{E}(\phi y_{t-1} + \epsilon_t)^2 \tag{2.1}$$

Since ϵ_t and y_{t-1} are independent and $\mathbf{E}y_{t-1}^2 = \mathbf{E}y_t^2$, Eq. (2.1) can be rewritten as

$$\gamma(0) = \phi^2\gamma(0) + \delta^2 \tag{2.2}$$

Therefore, $\gamma(0)$ has positive solution in Eq. (2.2) *i.i.f* $|\phi| < 1$, and $\gamma(0) = \frac{\delta^2}{1-\phi^2}$. Apparently the solution is unique.

Second, given $|\phi| < 1$, we discuss the autoregressive feature of AR(1). Suppose y_t starts from y_0, which may or may not be the stationary point of this AR(1). Denote *Lag operator* as L. When $t \to \infty$, y_t converges to

$$y_t = \lim_{t\to\infty}(\phi^t y_0 + \sum_{i=0}^{t-1}\phi^i\epsilon_{t-i}) = \lim_{t\to\infty}\phi^t y_0 + \frac{1}{1-\phi L}\epsilon_t = \frac{1}{1-\phi L}\epsilon_t \tag{2.3}$$

given

$$\lim_{t\to\infty}\phi^t y_0 = 0$$

regardless of y_0. Since $\epsilon_t \sim \mathbf{N}(0,\delta^2)$ is *i.i.d*, then it can be calculated that as $t \to \infty$, $\mathbf{E}y_t = 0$ and $\mathbf{E}y_t^2 = \frac{\delta^2}{1-\phi^2}$, the same as the stationary point discussed above. Therefore, the convergent point is the same as stationary point discussed above.

Another interpretation can be made as follows. Suppose that y_0 is the stationary point. Then y_t would always be at the stationary point, including when $t \to \infty$. Since y_t would converge to the same convergent point regardless of y_0, the stationary point is the same as the convergent point. Namely y_t would converge to the stationary point.

Remark. *Such a stationary point has significant similarity to a globally asymptotically stable equilibrium in control system.*