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Chapter 1

P. 2 Regarding the linear transformation, consider the closed convex set

$$C_1 = \{(x, y) \,|\, x > 0,\, y > 0,\, xy \geq 1\} \subset \mathbb{R}^2.$$

Its image under linear transformation $A = [1,\, 0]$ is $\{x \,|\, x > 0\}$, which is convex and open.

Regarding the vector sum, consider closed convex sets

$$C_1 = \{(x, y) \,|\, x > 0,\, y > 0,\, xy \geq 1\},\, C_2 = \{(x, 0) \,|\, x \leq 1\}.$$

Then their sum is $\{(x, y) \,|\, y > 0\}$, which is an open set. This is an example from [Note 1].

P. 3 $(\lambda_1 + \lambda_2)C \subset \lambda_1 C + \lambda_2 C$ is always true regardless whether C is convex.

P. 4 Such an r exists because it can be taken as $\min\{r_1, r_2\}$ where r_1 and r_2 are the radius of balls centered at x and y respectively, that are contained in C. The existence of r_1 and r_2 are asserted due to openness of C and $x, y \in C$.

P. 4

Lemma (P. 4)**.** *Let $P \subset \mathbb{R}^n$ be a polyhedral set that contains origin, and is given by $P = \cap_{j=1}^r F_j$, where $F_j = \{x \,|\, a_j' x \leq b_j\}$ is half space. In addition, P is a cone. Let P_j be the set $F_1 \cap \cdots \cap F_{j-1} \cap F_{j+1} \cap \cdots \cap F_r$, and for $j = 1, \ldots, r$, there exists some $x \in P_j$ that is not an element of P. Then $b_j = 0$ for $j = 1, \ldots, r$.*

Proof. Since $0 \in P$, then $b_j \geq 0$ for all j. In addition, since for $j = 1, \ldots, r$, there exists some $x \in P_j$ that is not an element of P, then $P \subset P_j$ and $P \neq P_j$. We prove the claim by contradiction given that $b_j \geq 0$. Assume $b_j > 0$. Then for all $y \in P_j \setminus P$, where $P_j \setminus P \neq \emptyset$, we have $y \in P_j$ and $a_j' y > b_j > 0$ since $y \notin F_j$. However, since $a_j' y > 0$, $b_j/(a_j' y) \in (0, 1)$ and $y \in P_j$, we have $a_j'(b_j y)/(a_j' y) \in F_j$, and $a_k'(b_j y)/(a_j' y) \leq b_k$ for all $k \neq j$, which means $(b_j y)/(a_j' y) \in P$. This is a contradiction with $y \notin P$ since P is a cone and $y = \lambda(b_j y)/(a_j' y)$ with $\lambda = (a_j' y)/b_j > 0$. Thus the assumption is false. The same arguments apply to every $j = 1, \ldots, r$. Thus, $b_j = 0$ for all j. □

P. 8 For a improper convex function $f : C \to [-\infty, \infty]$, it holds that $f(x) = -\infty$ $\forall x \in \mathrm{int}\big(\mathrm{dom}(f)\big)$. This statement can be found in 2.5, P. 41, [Rockafellar and Wets 98].

To see that, we first note that for the case $f(x) = \infty$, the statement holds. Otherwise, denote as $\bar{x}$ where $f(\bar{x}) = -\infty$, and we have that $\bar{x} \in \mathrm{dom}(f)$. Then for any $x \in \mathrm{int}\big(\mathrm{dom}(f)\big)$, there exists $r > 0$ such that the open ball centered at x with radius r contained in $\mathrm{dom}(f)$. Pick $z = x + \beta x - \beta \bar{x}$, where $0 < \beta < r/\|x - \bar{x}\|$, then we have $x = \alpha z + (1-\alpha)\bar{x}$, where $\alpha = 1/(1+\beta) \in (0,1)$. Since $z \in \mathrm{dom}(f)$, there exists ω such that $(z, w) \in \mathrm{epi}(f)$. By convexity of f, we have $f(x) \leq \alpha w + (1-\alpha)w'$ for all $\omega' \in \mathbb{R}$. Therefore, $f(x) = -\infty$.

P. 9 This definition is given with the understanding that for any subset $A \subset [-\infty, \infty]$, the infimum of $A \cup \{-\infty\}$, namely $\inf(A \cup \{-\infty\})$, is $-\infty$; and the infimum $\inf\{\infty\}$ is ∞. Similarly, $\sup(A \cup \{\infty\}) = \infty$, and $\sup\{-\infty\} = -\infty$. Note that to have objects such as $\sup \mathbb{R}$ or $\sup\{\infty\}$ defined, we are operating on the ordered set $[-\infty, \infty]$, namely, for set $Y \subset [-\infty, \infty]$, $\sup Y$ is defined as the greatest element in $[-\infty, \infty]$ such that it is no less than all $y \in Y$.

P. 11 The inverse is not true, that is, given a closed function $f : X \to [-\infty, \infty]$, its effective domain $\mathrm{dom} f$ can be open. One such example is $f(x) = 1/x$ defined over $(0, \infty)$.

On the other hand, if a function $f : X \to [-\infty, \infty]$ is closed, its extension to any $\overline{X}$, denoted as $\bar{f}$, such that $X \subset \overline{X}$ and

$$\bar{f}(x) = \begin{cases} f(x), & x \in X, \\ \infty, & x \in \overline{X} \setminus X, \end{cases}$$

is also closed.

P. 13

Lemma (1, P. 13)**.** *Let* $\{a_n\}, \{b_n\} \subset (-\infty, \infty]$ *be two convergent sequence with both limits* $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$ *in* $(-\infty, \infty]$*. It holds that*

$$\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \lim_{n\to\infty} (a_n + b_n). \tag{1}$$

Proof. Denote $a = \lim_{n\to\infty} a_n$ $b = \lim_{n\to\infty} b_n$. If $a = b = \infty$, then we have $\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \infty$. In addition, for any $k \in \mathbb{R}$, there exists N_a and N_b such that $a_n > k/2$ for all $n > N_a$ and $b_n > k/2$ for all $n > N_b$. Therefore, $a_n + b_n > k$ for all $n > \max\{N_a, N_b\}$. Per definition, this means that $\lim_{n\to\infty}(a_n + b_n) = \infty$.

Otherwise, without loss of generality, assume $a \in \mathbb{R}$. Then there exists N such that $\{a_n\}_{n=N}^{\infty} \subset \mathbb{R}$. Since sequences $\{a_n\}_{n=N}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ has the same limits, and so do $\{b_n\}_{n=N}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, and $\{a_n + b_n\}_{n=N}^{\infty}$ and $\{a_n + b_n\}_{n=1}^{\infty}$, then we work with $\{a_n\}_{n=N}^{\infty}$, $\{b_n\}_{n=N}^{\infty}$, and $\{a_n + b_n\}_{n=N}^{\infty}$. Denote their limits as $\bar{a}$, $\bar{b}$, and $\overline{a+b}$. By [Abstract DP Note Lemma 2, P. 42], we have $\bar{a} + \bar{b} = \overline{a+b}$. Therefore, the desired relation follows. ∎

Note that the above results can be extended to any finitely many convergent sequences which are within $(-\infty, \infty]$ with their limits in $(-\infty, \infty]$.

Similarly, the above result would also hold if $\{a_n\}, \{b_n\} \subset [-\infty, \infty)$ are two convergent sequence with both limits $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$ in $[-\infty, \infty)$.

Lemma (2, P. 13)**.** *Given functions $f_i : \mathbb{R}^n \to (-\infty, \infty]$, $i = 1, 2, ..., m$, that are all closed, then the function $f : \mathbb{R}^{mn} \to (-\infty, \infty]$ given as*

$$f(x) = \sum_{i=1}^{m} f_i(x_i)$$

where $x = (x_1, ..., x_m) \in \mathbb{R}^{mn}$, is also closed.

Proof. We first show that for any $\{x_i^k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ that converges to $x_i \in \mathbb{R}^n$, the sequence $\{\inf_{\ell \geq k} f_i(x_i^\ell)\}_{k=1}^{\infty}$ and its limit are within $(-\infty, \infty]$. Since for all i, $f_i(x)$ is closed thus lower semicontinuous, then given arbitrary $x_i \in \mathbb{R}^n$, every sequence $\{x_i^k\} \subset \mathbb{R}^n$ that converges to x_i, we have $\liminf_{k\to\infty} f_i(x_i^k) \geq f_i(x_i) > -\infty$, and the sequence $\{\inf_{\ell \geq k} f_i(x_i^\ell)\}_{k=1}^{\infty}$ is monotonically increasing. If $f_i(x_i) \in \mathbb{R}$, then for a finite $\underline{f_i} < f_i(x_i)$, there exists some K such that for all $k \geq K$, $\inf_{\ell \geq k} f_i(x_i^\ell) \geq \underline{f_i}$. If $f_i(x_i) = \infty$, then the monotonically increasing sequence $\{\inf_{\ell \geq k} f_i(x_i^\ell)\}_{k=1}^{\infty}$ has limit ∞, which means there exists some K such that for all $k \geq K$, $\inf_{\ell \geq k} f_i(x_i^\ell) \geq \underline{f_i}$. In either case, we have $\min\{f_i(x_i^1), f_i(x_i^2), \ldots, f_i(x_i^{K-1}), \underline{f_i}\}$ as a lower bound of $\inf_{\ell \geq 0} f_i(x_i^\ell)$, which is finite.

Then we show that the function f is lower semicontinuous. For every $x = (x_1, \ldots, x_m) \in \mathbb{R}^{mn}$, we look at every sequence $\{x^k\}$ that converges to x with $x^k = (x_1^k, \ldots, x_m^k)$. It is obvious that $x_i^k \to x_i$. Due to lower semicontinuity of f_i, we have for every i, $f_i(x_i) \leq \liminf_{k\to\infty} f_i(x_i^k)$. By above arguments, we see that $\{\inf_{\ell \geq k} f_i(x_i^\ell)\}_{k=1}^{\infty}$ and $\liminf_{k\to\infty} f_i(x_i^k)$ are all within $(-\infty, \infty]$ for all i, and so the term $\sum_{i=1}^m \liminf_{k\to\infty} f_i(x_i^k)$ is properly defined. Therefore, we have

$$f(x) = \sum_{i=1}^{m} f_i(x_i) \leq \sum_{i=1}^{m} \liminf_{k\to\infty} f_i(x_i^k) = \lim_{k\to\infty} \big(\sum_{i=1}^{m} \inf_{\ell \geq k} f_i(x_i^\ell)\big) \tag{2}$$

where the last equality is due to [Lemma 1, P. 13] and the comments followed. Since for all i it holds that $\inf_{\ell \geq k} f_i(x_i^\ell) \leq f_i(x_i^j)$ for all $j \geq k$, then $\sum_{i=1}^m \inf_{\ell \geq k} f_i(x_i^\ell) \leq \sum_{i=1}^m f_i(x_i^j)$ for all $j \geq k$. Therefore, $\sum_{i=1}^m \inf_{\ell \geq k} f_i(x_i^\ell) \leq \inf_{\ell \geq k} \big(\sum_{i=1}^m f_i(x_i^\ell)\big) = \inf_{\ell \geq k} f(x^\ell)$. Clearly the sequence $\{\inf_{\ell \geq k} f(x^\ell)\}$ is monotonically increasing and thus convergent, then by [Abstract DP Note Lemma 1, P. 42], we have

$$\lim_{k\to\infty} \big(\sum_{i=1}^{m} \inf_{\ell \geq k} f_i(x_i^\ell)\big) \leq \liminf_{k\to\infty} f(x^k). \tag{3}$$

Combine Eqs. (2) and (3) and we get the desired result. □

Note that in the above proof, we use $\inf_{\ell \geq k} f_i(x_i^\ell)$, with the understanding that $\{f_i(x_i^\ell)\}_{\ell \geq k}$ is a subset of $\mathbb{R} \cup \{\infty\}$, or may even be $\{\infty\}$ in the event that $f_i(x_i^\ell) = \infty \ \forall \ell \geq k$. In either case, its infimum is considered well-defined, and $\inf\{\infty\} = \infty$, as is commented in [COT Note P. 9].

Lemma (3, P. 13)**.** *Given functions $f_i : \mathbb{R}^n \to (-\infty, \infty]$, $i = 1, 2, ..., m$, that are all convex, then the function $f : \mathbb{R}^{mn} \to (-\infty, \infty]$ given as*

$$f(x) = \sum_{i=1}^{m} f_i(x_i)$$

where $x = (x_1, ..., x_m) \in \mathbb{R}^{mn}$, is also convex.

Proof. Since $f_i : \mathbb{R}^n \to (-\infty, \infty]$ is convex, per definition, $\text{epi}(f_i)$ is convex, then one can show that $\forall x_1, x_2 \in \mathbb{R}^n$ and $\theta \in [0, 1]$, it holds that $f_i\big(\theta x_1 + (1 - \theta)x_2\big) \leq \theta f_i(x_1) + (1 - \theta) f_i(x_2)$. Then we have $\forall x, y \in \mathbb{R}^{mn}$ and $\theta \in [0, 1]$ where $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$, it holds that

$$\begin{aligned} f\big(\theta x + (1 - \theta)y\big) =& \sum_{i=1}^{m} f_i\big(\theta x_i + (1 - \theta)y_i\big) \\ \leq& \sum_{i=1}^{m} \theta f_i(x_i) + (1 - \theta) f_i(y_i) \\ =& \theta f(x) + (1 - \theta) f(y). \end{aligned}$$

Now we need to show that $\text{epi}(f)$ is convex. To see that, given $(x, w), (y, v) \in \text{epi}(f)$, we have $f\big(\theta x + (1-\theta)y\big) \leq \theta f(x) + (1-\theta) f(y) \leq \theta w + (1-\theta)v$. Therefore, $\text{epi}(f)$ is convex. □

P. 13 The proof here uses also Prop. A.2.2(d).

P. 17 Denote $g : (0, 1] \to \mathbb{R}$ as

$$g(\alpha) = \frac{f\big(x^* + \alpha(z - x^*)\big) - f(x^*)}{\alpha},$$

which is well-defined since $x^*, z \in C$ and C is convex. Then due to f differentiable at x^*, then for any sequence $\{\alpha_k\} \subset (0, 1]$ that converges to 0, it holds that $\lim_{k \to \infty} g(\alpha_k) = -\varepsilon < 0$ where $-\varepsilon = \nabla f(x^*)'(z - x)$. Then it holds that

$$\exists \delta \in (0, 1]\Big(\forall \alpha\big(\alpha \in (0, \delta] \implies g(\alpha) < -\varepsilon/2\big)\Big). \tag{4}$$

To prove this, we assume otherwise is true, which states

$$\forall \delta\Big(\delta \in (0, 1] \implies \exists \alpha \in (0, \delta]\big(g(\alpha) \geq -\varepsilon/2\big)\Big). \tag{5}$$

By this statement, we take $\delta_n = 1/n$ and there exists $\alpha_n \in (0, 1/n]$ such that $g(\alpha_n) \geq -\varepsilon/2$ and $\alpha_n \to 0$, which contradicts $g(\alpha) \downarrow -\varepsilon$. Therefore (5) is false and (4) is true.

P. 17 Denote $C_w = C \cap \{\|z - x\| \le \|z - w\|\}$, then we have $C_w \subset C$. Denote $\underline{f} = \inf_{x \in C_w} f(x)$. Then by the definition of infimum, $\underline{f}$ is also the infimum of C. Since the minimum is attained in C_w at x^*, then $x^* \in C_w \subset C$. Therefore, the minimum is attained in C.

P. 18 To see this, due to the assumption that $\nabla^2 f(x)$ is not positive semidefinite, then there exists a unitary vector u such that $u'\nabla^2 f(x)u < 0$. Since $\nabla^2 f(x)$ is continuous, which means

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j},\ i, j = 1, ..., n \tag{6}$$

are continuous for all i and j, then the function $g(x) = u'\nabla^2 f(x)u$, which is the weighted sum of (6) with weights $u_i u_j$, is also continuous. Therefore, $\exists \varepsilon > 0$ such that $g(x + \alpha\varepsilon u) < 0$ for all $\alpha \in [0, 1]$. Therefore, we set $z = \varepsilon u$ and we have $z'\nabla^2 f(x + \alpha z)z < 0$ for all $\alpha \in [0, 1]$.

P. 20 [COTe1] Ex 1.11 (b), P. 17. To see the equality hold, denote as S_1 and S_2 the following sets

$$\begin{aligned} S_1 &= \Big\{ \sum_{i \in I} \alpha_i (x_i - \bar{x}) \,\Big|\, x_i \in C,\ i \in I,\ \sum_{i \in I} \alpha_i = 1,\ I \text{ is a finite set} \Big\}, \\ S_2 &= \Big\{ \sum_{i \in I} \beta_i (x_i - \bar{x}) \,\Big|\, x_i \in C,\ i \in I,\ I \text{ is a finite set} \Big\}. \end{aligned}$$

Then for every $s \in S_1$, we see that $s \in S_2$. On the other hand, for every $s \in S_2$, we have $s = \sum_{i \in I} \beta_i (x_i - \bar{x}) = \sum_{i \in I} \beta_i (x_i - \bar{x}) + (1 - \sum_{i \in I} \beta_i)(\bar{x} - \bar{x})$, and therefore $s \in S_1$. So we have $S_1 = S_2$.

P. 20 [COTe1] Ex 1.11 (b), P. 17.

Lemma (COTe1, P. 17)**.** *Given a set X, for the subspace V spanned by X,*

$$V = \Big\{ \sum_{i \in I} \alpha_i x_i \,\Big|\, x_i \in X,\ \alpha_i \in \mathbb{R},\ \forall i \in I,\ I \textit{ is a finite set} \Big\}, \tag{7}$$

denote its dimension as m. Then there exists $x_1, ..., x_m \in X$ that are linearly independent, and forms a basis of V.

Proof. We first prove that there exists $x_1, ..., x_m \in X$ that are linearly independent. Without loss of generality, we assume the maximum number of vectors in X that are linearly independent are n, which is smaller than m, and the set of vectors as $x_1^*, ..., x_n^*$. Then all $x \in X$ can be written as linear combinations of $x_1^*, ..., x_n^*$ (otherwise, assume x cannot, then the set of vectors $x_1^*, ..., x_n^*, x$ are linearly independent, which contradicts the assumption). Since V has dimension m, then denote its basis as $s_1, ..., s_m$. Since $s_i \in V$, then $s_i = \sum_{\ell \in I} \beta_\ell^i x_\ell^i$. Since x_ℓ^i can be written as linear combinations of $x_1^*, ..., x_n^*$, by rewriting every

x^i_ℓ with x_1^*, ..., x_n^*, we get $s_i = \sum_{j=1}^n \alpha_j^i x_j^*$. Now for the basis set s_1, ..., s_m, we consider λ_1, ..., $\lambda_m \in \mathbb{R}$ such that $\sum_{i=1}^m \lambda_i s_i = 0$, then we have

$$\sum_{j=1}^{n} \sum_{i=1}^{m} \lambda_i \alpha_j^i x_j^* = 0.$$

Since x_1^*, ..., x_n^* are linearly independent, then $\sum_{i=1}^m \lambda_i \alpha_j^i = 0$ for $j = 1$, ..., n. Therefore, we get a set of linear equations $A\Lambda = 0$ where $\Lambda = (\lambda_1, ..., \lambda_m)$ and A is a $n \times m$ matrix with jith element of A as α_j^i. Then there would be nontrivial λ_1, ..., λ_m that fulfills $\sum_{i=1}^m \lambda_i s_i = 0$. Therefore, the assumption is false. If on the other hand, the maximum number n is bigger than m, it is a direct contradictions with the dimension m. Therefore, the proof is done. □

P. 24 To see this, consider vector z which is in $S_\alpha \cap \operatorname{aff}(C)$. Then z could be written as $z = x_\alpha + \delta u$ where $\delta \in (0, \alpha\varepsilon)$ and u is a unite length vector. Since $x_\alpha \in C$, the affine hull of C can be written as $x_\alpha + M$ where M is the subspace parallel to $\operatorname{aff}(C)$ [refer to comment on P. 229]. Since $z \in \operatorname{aff}(C)$, then $u \in M$. Since $x \in C$, we also have $\operatorname{aff}(C) = x + M$. Therefore, $x + \frac{\delta}{\alpha} u \in x + M = \operatorname{aff}(C)$. In addition, $x + \frac{\delta}{\alpha} u \in S$ in view of definition of δ. Therefore, $z = \alpha(x + \frac{\delta}{\alpha} u) + (1 - \alpha)\bar{x}$.

P. 25 $X \subset C$ since due to assumption that $0 \in C$, then any $x \in X$ can be interpreted as $x = \sum_{i=1}^m \alpha_i z_i + (1 - \sum_{i=1}^m \alpha_i)0$, which is a convex combination of elements in C.

P. 26

Lemma (1, P. 26)**.** *Let $C \subset \mathbb{R}^n$ be a nonempty set and $\bar{x} \in C$. Then it holds that* $\operatorname{aff}(C) = \bar{x} + \operatorname{aff}(C - \bar{x})$.

Proof. It is clear that $0 \in C - \bar{x}$. In addition, denote $\operatorname{aff}(C) = \bar{x} + S$, where by the arguments given in the proof of [COTe1] Ex 1.11(b), P. 17, we have

$$S = \Big\{ \sum_{i \in I} \alpha_i (x_i - \bar{x}) \,\Big|\, x_i \in C,\, i \in I,\, \sum_{i \in I} \alpha_i = 1,\, I \text{ is a finite set} \Big\}. \quad (8)$$

On the other hand, by the conclusion of [COTe1] Ex 1.11 (b), P. 17, we have

$$\operatorname{aff}(C - \bar{x}) = \Big\{ \sum_{i \in I} \alpha_i (x_i - \bar{x}) \,\Big|\, x_i \in C,\, i \in I,\, \sum_{i \in I} \alpha_i = 1,\, I \text{ is a finite set} \Big\}. \quad (9)$$

Clearly, we have $\operatorname{aff}(C - \bar{x}) = S$. Therefore, we have $\operatorname{aff}(C) = \bar{x} + \operatorname{aff}(C - \bar{x})$. □

Lemma (2, P. 26)**.** *Let $C \subset \mathbb{R}^n$ be a nonempty set, and $y \in \mathbb{R}^n$. Then it holds that* $\operatorname{aff}(C + y) = y + \operatorname{aff}(C)$.

Proof. Since C is nonempty, denote as x an element of C. Then we have $(x+y) \in (y+C)$. By [Lemma 1, P. 26], we have $\operatorname{aff}(C) = x + \operatorname{aff}(C-x)$, and $\operatorname{aff}(C+y) = x+y+\operatorname{aff}\big(C+y-(x+y)\big) = x+y+\operatorname{aff}\big(C-y\big)$. Therefore, we have $\operatorname{aff}(C+y) = y + \operatorname{aff}(C)$. □

Lemma (3, P.26)**.** *Given a nonempty convex set $C \subset \mathbb{R}^n$, if $\bar{x} \in \operatorname{ri}(C)$, then for every $y \in \mathbb{R}^n$, $\bar{x}+y \in \operatorname{ri}(y+C)$.*

Proof. Since we have $\bar{x} \in \operatorname{ri}(C)$, then $\bar{x}+y \in (y+C)$ and $\operatorname{aff}(C+y) = y+\operatorname{aff}(C)$ by [Lemma 2, P. 26]. Then there exists an open ball centered at $\bar{x}$ with radius ε, which we denote as $B_\varepsilon(\bar{x})$, such that $B_\varepsilon(\bar{x}) \cap \operatorname{aff}(C) \subset C$. On the other hand, we have $B_\varepsilon(\bar{x}+y) = B_\varepsilon(\bar{x}) + y$, $\operatorname{aff}(C+y) = \operatorname{aff}(C)+y$ as given above, then $B_\varepsilon(\bar{x}+y) \cap \operatorname{aff}(C+y) \subset y + C$, which means $\bar{x}+y \in \operatorname{ri}(C+y)$. □

P. 28

Lemma (P. 28)**.** *Let $C \subset \mathbb{R}^n$ be a nonempty convex set. Then it holds that* $\operatorname{ri}\big(\operatorname{ri}(C)\big) = \operatorname{ri}(C)$.

Proof. By [COT Prop. 1.3.5 (a), P. 28], we have $\operatorname{cl}\big(\operatorname{ri}(C)\big) = \operatorname{cl}(C)$. Namely, C and $\bar{C} = \operatorname{ri}(C)$ have the same closure. Then by [COT Prop. 1.3.5 (c), P. 28], they have the same relative interior, viz., $\operatorname{ri}\big(\operatorname{ri}(C)\big) = \operatorname{ri}(C)$. □

P. 29

Lemma (P. 29)**.** *Let $X \subset \mathbb{R}^n$ be a nonempty set and A an $m \times n$ matrix. Then it holds that* $\operatorname{cl}\big(A \cdot \operatorname{cl}(X)\big) = \operatorname{cl}(A \cdot X)$.

Proof. In [COT Prop. 1.3.6 (b), P. 29], it is proved that $A \cdot \operatorname{cl}(X) \subset \operatorname{cl}(A \cdot X)$. Therefore, it holds that $\operatorname{cl}\big(A \cdot \operatorname{cl}(X)\big) \subset \operatorname{cl}(A \cdot X)$. To see the reverse, for every $y \in \operatorname{cl}(A \cdot X)$, there exists $\{y_k\} \subset A \cdot X$ such that $y_k \to y$. However, we have $A \cdot X \subset A \cdot \operatorname{cl}(X)$. Therefore, $\{y_k\} \subset A \cdot \operatorname{cl}(X)$ and $y \in \operatorname{cl}\big(A \cdot \operatorname{cl}(X)\big)$. □

P. 31

Lemma (1, P. 31)**.** *Let X_1, X_2 be two nonempty sets of $\mathbb{R}^n$. Then it holds that* $\operatorname{aff}(X_1 \times X_2) = \operatorname{aff}(X_1) \times \operatorname{aff}(X_2)$.

Proof. Denote the dimensions of $\operatorname{aff}(X_1)$, $\operatorname{aff}(X_2)$ as m_1 and m_2 respectively. Then by [COTe1 Ex 1.11 (b), P. 17], we have, for some $x_1^1, \ldots, x_{m_1}^1, \bar{x}^1 \in X_1$, and $x_1^2, \ldots, x_{m_2}^2, \bar{x}^2 \in X_2$,

$$
\begin{aligned}
\operatorname{aff}(X_1) &= \Big\{ y \,\Big|\, y = \sum_{i=1}^{m_1} \alpha_i^1 (x_i^1 - \bar{x}^1) + \bar{x}^1 \Big\}, \\
\operatorname{aff}(X_2) &= \Big\{ y \,\Big|\, y = \sum_{i=1}^{m_2} \alpha_i^2 (x_i^2 - \bar{x}^2) + \bar{x}^2 \Big\}.
\end{aligned}
\tag{10}
$$

For any $(y^1, y^2) \in X_1 \times X_2$, we have $(y^1, y^2) \in \operatorname{aff}(X_1) \times \operatorname{aff}(X_2)$. In addition, $\operatorname{aff}(X_1) \times \operatorname{aff}(X_2)$ is affine (one can verify this by using the definition of affine set). Therefore, $\operatorname{aff}(X_1 \times X_2) \subset \operatorname{aff}(X_1) \times \operatorname{aff}(X_2)$.

As for the reverse direction, for any $(y^1, y^2) \in \operatorname{aff}(X_1) \times \operatorname{aff}(X_2)$, we have, by (10),

$$
\begin{aligned}
y^1 &= \sum_{i=1}^{m_1} \beta_i (x_i^1 - \bar{x}^1) + \bar{x}^1, \\
y^2 &= \sum_{i=1}^{m_2} \gamma_i (x_i^2 - \bar{x}^2) + \bar{x}^2.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
(y^1, y^2) &= \sum_{i=1}^{m_1} \beta_i \big((x_i^1, \bar{x}^2) - (\bar{x}^1, \bar{x}^2) \big) + \sum_{i=1}^{m_2} \gamma_i \big((\bar{x}^1, x_i^2) - (\bar{x}^1, \bar{x}^2) \big) + (\bar{x}^1, \bar{x}^2) \\
&= \sum_{i=1}^{m_1} \beta_i (x_i^1, \bar{x}^2) + \sum_{i=1}^{m_2} \gamma_i (\bar{x}^1, x_i^2) + \big(1 - \sum_{i=1}^{m_1} \beta_i - \sum_{i=1}^{m_2} \gamma_i \big) (\bar{x}^1, \bar{x}^2),
\end{aligned}
$$

which is affine combination of elements in $X_1 \times X_2$. Therefore, the proof is complete. □

Lemma (2, P. 31)**.** *Let S_i, T_i be two sequence of sets indexed by I. Then we have $\prod_{i \in I} S_i \cap \prod_{i \in I} T_i = \prod_{i \in I} (S_i \cap T_i)$.*

Proof. $x \in \prod_{i \in I} S_i \cap \prod_{i \in I} T_i$ implies that

$$
\forall i (i \in I \implies x_i \in S_i) \wedge \forall i (i \in I \implies x_i \in T_i).
$$

Applying the second inference rule to the first, we get

$$
\forall i (i \in I \implies x_i \in S_i \wedge x_i \in T_i),
$$

which per definition of set intersection, we have

$$
\forall i (i \in I \implies x_i \in S_i \cap T_i),
$$

and we have $x \in \prod_{i \in I} (S_i \cap T_i)$. The reverse can be similarly proven. □

Lemma (3, P. 31)**.** *Let C_1, C_2 be two nonempty convex sets of $\mathbb{R}^n$. Then it holds that* $\operatorname{ri}(C_1 \times C_2) = \operatorname{ri}(C_1) \times \operatorname{ri}(C_2)$.

Proof. Here we apply the infinity norm. For $x_1 \in \operatorname{ri}(C_1)$, $x_2 \in \operatorname{ri}(C_2)$, we have $B_{\varepsilon_1}(x_1) \cap \operatorname{aff}(C_1) \subset C_1$, and $B_{\varepsilon_2}(x_2) \cap \operatorname{aff}(C_2) \subset C_2$, where $B_{\varepsilon_1}(x_1)$ is a ball centered at x_1 with radius ε_1 measured by infinity norm. Denote $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then we have $B_\varepsilon\big((x_1, x_2)\big) \subset B_{\varepsilon_1}(x_1) \times B_{\varepsilon_2}(x_2)$. Then we have

$$
\begin{aligned}
B_\varepsilon\big((x_1, x_2)\big) \cap \mathrm{aff}(C_1 \times C_2) \subset & B_{\varepsilon_1}(x_1) \times B_{\varepsilon_2}(x_2) \cap \mathrm{aff}(C_1 \times C_2) \\
= & B_{\varepsilon_1}(x_1) \times B_{\varepsilon_2}(x_2) \cap \mathrm{aff}(C_1) \times \mathrm{aff}(C_2) \\
= & \big(B_{\varepsilon_1}(x_1) \cap \mathrm{aff}(C_1)\big) \times \big(B_{\varepsilon_2}(x_2) \times \mathrm{aff}(C_2)\big) \\
\subset & C_1 \times C_2,
\end{aligned}
$$

where the first equality is due to [Lemma 1, P. 31], and the second equality is due to [Lemma 2, P. 31]. Therefore, $\mathrm{ri}(C_1) \times \mathrm{ri}(C_2) \subset \mathrm{ri}(C_1 \times C_2)$. For the reverse direction, we need to show that given $B_\varepsilon\big((x_1, x_2)\big) \cap \mathrm{aff}(C_1 \times C_2) \subset C_1 \times C_2$, $x_1 \in \mathrm{ri}(C_1)$, and $x_2 \in \mathrm{ri}(C_2)$. The proof is entirely similar. □

P. 31 Since $\{x_k^1\}$ and $\{x_k^2\}$ are both bounded sequences in $\mathbb{R}^n$, then $\{(x_k^1, x_k^2)\}$ is a bounded sequence in $\mathbb{R}^{2n}$ (most easily seen via infinity norm). Therefore, $\{(x_k^1, x_k^2)\}$ has a convergent subsequence.

P. 32

Lemma (1, P. 32)**.** *Given a set X, denote as V a subspace spanned by X with dimension m. Let $\{z_1, \ldots, z_n\} \subset X$ be a set of linearly independent elements and $n < m$. Then there exists a basis of V which includes $\{z_1, \ldots, z_n\}$.*

Proof. By [Lemma COTe1, P. 17], there exists $\{x_1, \ldots, x_m\} \subset X$ that form a basis of V. Then every element of $\{x_1, \ldots, x_m\}$ can be either independent or not with $\{z_1, \ldots, z_n\}$. Taking those that are independent with $\{z_1, \ldots, z_n\}$ and form the set $\{x^1, \ldots, x^k\}$. We argue that $n + k = m$. Assume otherwise, then if $n + k > m$, it is a direct contradiction. If $n + k < m$, then since any element of V can be written as linear combination of elements in $\{x_1, \ldots, x_m\} \setminus \{x^1, \ldots, x^k\}$ and $\{x^1, \ldots, x^k\}$, and elements in $\{x_1, \ldots, x_m\} \setminus \{x^1, \ldots, x^k\}$ are linear combination of $\{z_1, \ldots, z_n\}$. Therefore, every element in V can be written as linear combination of $\{z_1, \ldots, z_n, x^1, \ldots, x^k\}$. Therefore, it is a basis and $n + k < m$ results in a contradiction with the dimension. By the same argument, we can see that the set $\{z_1, \ldots, z_n, x^1, \ldots, x^k\}$ is a basis of V. □

Lemma (2, P. 32)**.** *Given nonempty convex sets C_1 and C_2, it holds that* $\mathrm{aff}(C_1 \cap C_2) \subset \mathrm{aff}(C_1) \cap \mathrm{aff}(C_2)$.

Proof. When $C_1 \cap C_2 = \emptyset$, the relation clearly holds. Otherwise, for any $x \in C_1 \cap C_2$, it holds that $x \in \mathrm{aff}(C_1)$ and $x \in \mathrm{aff}(C_2)$, and therefore $x \in \mathrm{aff}(C_1) \cap \mathrm{aff}(C_2)$, namely $C_1 \cap C_2 \subset \mathrm{aff}(C_1) \cap \mathrm{aff}(C_2)$ Since intersection of affine sets are affine. Therefore, $\mathrm{aff}(C_1) \cap \mathrm{aff}(C_2)$ is an affine set which contains $C_1 \cap C_2$. Therefore, $\mathrm{aff}(C_1 \cap C_2) \subset \mathrm{aff}(C_1) \cap \mathrm{aff}(C_2)$. □

An alternative proof can be given as follows.

Proof. When $C_1 \cap C_2 = \emptyset$, the relation clearly holds. Otherwise, denote as $\bar{x}$ some element of $C_1 \cap C_2$. Denote as m the dimension of $\mathrm{aff}(C_1 \cap C_2)$. Then by [COTe1 Ex 1.11 (b), P. 17], we have for some $x_1, \ldots, x_m \in C_1 \cap C_2$, it holds that

$$\mathrm{aff}(C_1 \cap C_2) = \Big\{ y \,\Big|\, y = \sum_{i=1}^{m} \alpha_i (x_i - \bar{x}) + \bar{x} \Big\}. \tag{11}$$

As for $\mathrm{aff}(C_1)$, by [COTe1 Ex 1.11 (b), P. 17] and [Lemma 1, P. 32], it can be written as

$$\mathrm{aff}(C_1) = \Big\{ y \,\Big|\, y = \sum_{i=1}^{m} \alpha_i (x_i - \bar{x}) + \sum_{i=m+1}^{m^1} \beta_i^1 (z_i^1 - \bar{x}) + \bar{x} \Big\}, \tag{12}$$

where z_i^1's are elements in C_1 and m^1 is the dimension of $\mathrm{aff}(C_1)$. Similarly, we have

$$\mathrm{aff}(C_2) = \Big\{ y \,\Big|\, y = \sum_{i=1}^{m} \alpha_i (x_i - \bar{x}) + \sum_{i=m+1}^{m^2} \beta_i^1 (z_i^2 - \bar{x}) + \bar{x} \Big\}. \tag{13}$$

Clearly, we have $\mathrm{aff}(C_1 \cap C_2) \subset \mathrm{aff}(C_1) \cap \mathrm{aff}(C_2)$. □

From Eqs. (11), (12), and (13), we can see that the reason that $\mathrm{aff}(C_1 \cap C_2)$ and $\mathrm{aff}(C_1) \cap \mathrm{aff}(C_2)$ may not be equal is that the set of vectors $\{z_i^1 - \bar{x}\}_{i=m+1}^{m^1}$ and $\{z_i^2 - \bar{x}\}_{i=m+1}^{m^2}$ may be linearly dependent.

By [Lemma 2, P. 32], we can have an alternative proof for $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \subset \mathrm{ri}(C_1 \cap C_2)$. For $x \in \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$, we have $B_{\varepsilon_1}(x) \cap \mathrm{aff}(C_1) \subset C_1$ and $B_{\varepsilon_2}(x) \cap \mathrm{aff}(C_2) \subset C_2$ where the balls are measured by the infinity norm. Then denote $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, and we have

$$\begin{aligned} B_\varepsilon(x) \cap \mathrm{aff}(C_1 \cap C_2) \subset & B_\varepsilon(x) \cap \mathrm{aff}(C_1) \cap \mathrm{aff}(C_2) \\ \subset & B_\varepsilon(x) \cap \mathrm{aff}(C_1) \\ \subset & C_1. \end{aligned}$$

The relation $B_\varepsilon(x) \cap \mathrm{aff}(C_1 \cap C_2) \subset C_2$ can be proved the same way. Therefore, we have $B_\varepsilon(x) \cap \mathrm{aff}(C_1 \cap C_2) \subset C_1 \cap C_2$.

P. 32

Lemma (3, P. 32)**.** *Given nonempty convex sets C_1 and C_2, if $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_1) \neq \emptyset$, then it holds that $\mathrm{aff}(C_1 \cap C_2) = \mathrm{aff}(C_1) \cap \mathrm{aff}(C_2)$.*

Proof. In view of [Lemma 2, P. 32], we only need to show that $\mathrm{aff}(C_1) \cap \mathrm{aff}(C_2) \subset \mathrm{aff}(C_1 \cap C_2)$. Since $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ is nonempty, denote $x \in \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$. Then by [COTe1 Ex 1.24(a), P. 32], for any $y \in \mathrm{aff}(C_1) \cap \mathrm{aff}(C_2)$, there exists $\gamma_1, \gamma_2 > 0$ such that $z_1 = x + \gamma_1(x - y) \in C_1$ and $z_2 = x + \gamma_2(x - y) \in C_2$. Without loss of generality, we assume $\gamma_1 \leq \gamma_2$. Then in view of the convexity of C_2, we see that z_1 is convex combination of x and z_2 and therefore $z_1 \in C_2$, namely

$z_1 \in C_1 \cap C_2$. Then we have $z_1, x \in C_1 \cap C_2$ and y is affine combination of z_1 and x. Therefore, any affine set containing $C_1 \cap C_2$ contains y, which means $y \in \text{aff}(C_1 \cap C_2)$. □

P. 32 Here we fill in some details. By Prolongation Lemma, there exist $\gamma_1, \gamma_2 > 0$, such that $z_1 \in C_1$ and $z_2 \in C_2$ where $z_1 = x + \gamma_1(x - y)$ and $z_2 = x + \gamma_2(x - y)$. With out loss of generality, assume $\gamma_1 \leq \gamma_2$. Then due to convexity of C_2, we can see that $z_1 \in C_2$. Therefore, $z_1 \in C_1 \cap C_2$.

P. 33 We now give two examples to see that when $A^{-1} \cdot \text{ri}(C)$ is empty, the relation does not hold.

(1) Consider matrix A and set C as

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; C = \{(x, y) \mid x \geq 0, \, y = 0\}.$$

Then we have $A^{-1} \cdot \text{ri}(C) = \emptyset$, and $\text{ri}(A^{-1} \cdot C) = \{(0, 0)\} \neq A^{-1} \cdot \text{ri}(C)$.

(2) Consider matrix A and set C as

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; C = \{(x, y) \mid x > 0, \, y = 0\}.$$

Then we have $A^{-1} \cdot \text{ri}(C) = \emptyset$, and $\text{cl}(A^{-1} \cdot C) = \emptyset \neq \{(0, 0)\} = A^{-1} \cdot \text{cl}(C)$.

P. 34 Given an affine set S, its affine hull $\text{aff}(S)$ is S. Therefore, its relative interior $\text{ri}(S)$ is S as well, due to the definition.

P. 35 Denote the projection mapping as T, e.g., $T(x, y) = x$. Then $\text{ri}(D) = \text{ri}(T \cdot C)$. By [COT Prop. 1.3.6 (a), P. 29], we have $\text{ri}(D) = T \cdot \text{ri}(C)$, which means, $\forall x$,

$$x \in \text{ri}(D) \iff \exists y\big((x, y) \in \text{ri}(C)\big). \tag{14}$$

On the other hand, we also have, per the definition of nonempty set, that $\forall x$,

$$\exists y\big((x, y) \in \text{ri}(C)\big) \iff M_x \cap \text{ri}(C) \neq \emptyset. \tag{15}$$

Then we have, $\forall (x, y)$,

$$\begin{aligned} (x, y) \in \text{ri}(C) &\implies M_x \cap \text{ri}(C) \neq \emptyset \\ &\iff \exists y'\big((x, y') \in \text{ri}(C)\big) \\ &\iff x \in \text{ri}(D). \end{aligned} \tag{16}$$

Note that in (16), asserting $(x, y) \in \text{ri}(C)$ contains more information than $M_x \cap \text{ri}(C) \neq \emptyset$ therefore implies $M_x \cap \text{ri}(C) \neq \emptyset$, since not only have we asserted that $M_x \cap \text{ri}(C)$ is nonempty, but also we have given an element (x, y) of the set, about which (here 'which' refers to the element (x, y)) we may say more if we would like to.

Since via above derivation, we have $(x, y) \in \mathrm{ri}(C) \implies x \in \mathrm{ri}(D) \wedge (x, y) \in M_x \cap \mathrm{ri}(C)$, then it would imply that $\exists x' \big(x' \in \mathrm{ri}(D) \wedge (x, y) \in M_{x'} \cap \mathrm{ri}(C)\big)$ (similar to the derivation in (16) via arguing that the set of x' is nonempty). Via above derivation, we establish $\mathrm{ri}(C) \subset \cup_{x \in \mathrm{ri}(D)} \big(M_x \cap \mathrm{ri}(C)\big)$. The reverse direction is easy to show.

In addition, note that the strict orthodox way to define $T \cdot \mathrm{ri}(C)$ would be, for $x \in T \cdot \mathrm{ri}(C)$,

$$\begin{aligned}\exists x' \exists y' \big((x', y') \in \mathrm{ri}(C) \wedge T(x', y') = x\big) &\iff \exists x' \exists y' \big((x', y') \in \mathrm{ri}(C) \wedge x' = x\big) \\ &\iff \exists x' \exists y' \big((x, y') \in \mathrm{ri}(C) \wedge x' = x\big) \\ &\iff \exists y' \big((x, y') \in \mathrm{ri}(C)\big) \wedge \exists x' (x' = x),\end{aligned}$$

where the first $\iff$ is due to definition of T, the second is per the primitive $=$, and the third is by rules of bound quantifiers and their ranges. Since it is always true that

$$\forall x \big(\exists x' (x' = x)\big),$$

so the orthodox definition is simplified to be the right-hand side of (14). Similar arguments go for (15).

P. 36 If the point x is origin o, then by setting $\alpha_i = 1/2^n$, the statement holds. Otherwise $x \neq o$ and denote $x = (x^1, \ldots, x^n)$ where $|x^i| \leq 1$. Then we consider points on the line connecting x and o, namely $\{\alpha(x^1, \ldots, x^n) \mid \alpha \in \mathbb{R}\}$. Without loss of generality, we assume that $1 \in \arg\max_i |x_i|$, then we have

$$(x^1, x^2, \ldots, x^n) = \beta\big(1, \frac{x^2}{x^1}, \ldots, \frac{x^n}{x^1}\big) + (1 - \beta)\big(-1, -\frac{x^2}{x^1}, \ldots, -\frac{x^n}{x^1}\big),$$

where $\beta = (1 + x^1)/2 \in [0, 1]$. Therefore, we see that x is a convex combination of $(1, x^2/x^1, \ldots, x^n/x^1)$ and $-(1, x^2/x^1, \ldots, x^n/x^1)$. Now if we focus on $(1, x^2/x^1, \ldots, x^n/x^1)$, we see that if all the terms $x^i/x^1 \in \{1, -1\}$, we are done; otherwise, repeat above procedure with X replaced by $X^1 = \{x \mid \|x\|_\infty \leq 1,\ x^1 = 1\}$, and o replaced by $o^1 = (1, 0, \ldots, 0)$. Eventually, the term x would be written as convex combination of e_i's.

Intuitively, what we have done above is to first construct the line connecting x and o. This line would intersect with facets (including edges) of X with two points. Those two points can form a convex combination of x. Either one of the two points can then be written as convex combination of corners of the corresponding facets by repeating the above procedures.

P. 36 This is without loss of generality since for any $\{x_k\}$ that $x_k \to 0$, there exists K such that $x_k \in X$ for all $k \geq K$. In addition, when $x_k = 0$, we can define y_k and z_k randomly, such as $y_k = e_1$ and $z_k = -y_k$, and the followup proof could be carried out identically.

P. 37 Note that the relative interior is the interior itself.

P. 37 Another way for construction is as follows. If the sequence $\{x_k\}$ is identically $\bar{x}$, then the followup discussion trivially hold. Otherwise, assume first that $\bar{x} = \inf C$. Then for sequence $\{x_k\} \subset C$ that converges to $\bar{x}$, we have $x_k \geq \bar{x}$. In addition, we see that $\{x_k\}$ is bounded and $\bar{x}_0 = \sup\{x_k\}$ is a limit point of C. Since C is closed, we have $\bar{x}_0 \in C$. In addition, since $\{x_k\}$ is not identically $\bar{x}$, then there exists $x_k > \bar{x}$, which implies that $\bar{x}_0 > \bar{x}$. Therefore, for all x_k, it holds that

$$x_k = \alpha_k \bar{x}_0 + (1 - \alpha_k)\bar{x},$$

where $\alpha_k \in [0, 1]$. When $\bar{x} = \sup C$, entirely similar arguments could be applied.

P. 38 Here we emphasize that for a given function $f : X \to [-\infty, \infty]$, we define its closure $\text{cl}\, f$ as

$$(\text{cl}\, f)(x) = \inf \big\{ w \,\big|\, (x, w) \in \text{cl}\big(\text{epi}(f)\big) \big\}, \ \forall x \in \mathbb{R}^n. \tag{17}$$

Lemma (1, P. 38)**.** *Given a function $f : X \to [-\infty, \infty]$, the closure of its epigraph $\text{cl}\big(\text{epi}(f)\big)$ is a legitimate epigraph.*

Proof. We prove the above claim by showing that $\text{cl}\big(\text{epi}(f)\big) = \text{epi}(\text{cl}\, f)$, where $\text{cl}\, f$ is the closure of f. For every $(x, w') \in \text{cl}\big(\text{epi}(f)\big)$, it holds that $(\text{cl}\, f)(x) \leq w'$, and therefore $(x, w') \in \text{epi}(\text{cl}\, f)$.

Now we need to prove $\text{epi}(\text{cl}\, f) \subset \text{cl}\big(\text{epi}(f)\big)$ and we use infinity norm to measure the distance. For every $(x, w) \in \text{epi}(\text{cl}\, f)$, $\infty > (\text{cl}\, f)(x) \geq -\infty$. If $(\text{cl}\, f)(x) > -\infty$, then for all positive integer k, there exists $(x, w_k) \in \text{cl}\big(\text{epi}(f)\big)$ such that $w_k < (\text{cl}\, f)(x) + 1/k$. Since $(x, w_k) \in \text{cl}\big(\text{epi}(f)\big)$, there exists $(x_k, w_k^k) \in \text{epi}(f)$ that is in the $1/k$ neighborhood of (x, w_k). Therefore, $(x_k, w_k^k) \in \text{epi}(f)$ is in the $2/k$ neighborhood of $(x, (\text{cl}\, f)(x))$, and we can see that $\{x_k, w_k^k\} \subset \text{epi}(f)$ converges to $(x, (\text{cl}\, f)(x))$. Therefore, $(x, (\text{cl}\, f)(x)) \in \text{cl}\big(\text{epi}(f)\big)$. Denote $w - (\text{cl}\, f)(x)$ as δ. Then we can see that $\{x_k, w_k^k + \delta\} \subset \text{epi}(f)$ and converges to (x, w). If $(\text{cl}\, f)(x) = -\infty$, then there exists $(x, w') \in \text{cl}\big(\text{epi}(f)\big)$ such that $w' < w$. Then there exists $\{x_k, w'_k\} \subset \text{epi}(f)$ that converges to (x, w'). Then $\{x_k, w'_k + w - w'\} \subset \text{epi}(f)$ and converges to (x, w), which concludes the proof. □

P. 38 The convex closure of a function $f : X \to [-\infty, \infty]$ is defined as

$$(\check{\text{cl}}\, f)(x) = \inf \big\{ w \,\big|\, (x, w) \in \text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big) \big\}, \ \forall x \in \mathbb{R}^n. \tag{18}$$

We have the following lemmas hold.

Lemma (2, P. 38)**.** *Given a function $f : X \to [-\infty, \infty]$, if $(x, w) \in \text{conv}\big(\text{epi}(f)\big)$, then it holds that $(x, w') \in \text{conv}\big(\text{epi}(f)\big)$ for all $w' \geq w$.*

Proof. Due to Caratheodory's theorem [COT Prop. 1.2.1, P. 20], $(x, w) \in \text{conv}(\text{epi}(f))$ implies that $(x, w) = \sum_{i \in I} \alpha_i (x_i, w_i)$ where $(x_i, w_i) \in \text{epi}(f)$ and $\sum_{i \in I} \alpha_i = 1$. Denote $w' - w$ as β. Then $(x_i, w_i + \beta) \in \text{epi}(f)$. Therefore, $\sum_{i \in I} \alpha_i (x_i, w_i + \beta) \in \text{conv}(\text{epi}(f))$, which concludes the proof. □

Lemma (3, P. 38)**.** *Given a function* $f : X \to [-\infty, \infty]$*, the closure of the convex hull of its epigraph* $\text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big)$ *is a legitimate epigraph.*

Proof. We prove this statement via showing that $\text{epi}(\check{\text{cl}}\, f) = \text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big)$, where $\check{\text{cl}}\, f$ is defined in (18).

First, to show $\text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big) \subset \text{epi}(\check{\text{cl}}\, f)$, we note that for $(x, w) \in \text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big)$, we have $(\check{\text{cl}}\, f)(x) \leq w$, which implies that $(x, w) \in \text{epi}(\check{\text{cl}}\, f)$.

Then we show the reverse direction. For every $(x, w) \in \text{epi}(\check{\text{cl}}\, f)$, we have $(\check{\text{cl}}\, f)(x) \leq w$. If $(\check{\text{cl}}\, f)(x) > -\infty$, then there exists $\{(x, w_k)\} \subset \text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big)$ such that $w_k \to (\check{\text{cl}} f)(x)$. Then for every k, there exists $(x_k, w_k^k) \in \text{conv}\big(\text{epi}(f)\big)$ such that $\|x_k - x\|_\infty \leq 1/k$, and $\|w_k^k - w_k\|_\infty \leq 1/k$. Therefore, we see that $(x_k, w_k^k) \to \big(x, (\check{\text{cl}}\, f)(x)\big)$, which implies that $\big(x, (\check{\text{cl}}\, f)(x)\big) \in \text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big)$. Denote $w - (\check{\text{cl}}\, f)(x)$ as β. By [Lemma 2, P. 38], we have that $\{(x_k, w_k^k + \beta)\} \subset \text{conv}\big(\text{epi}(f)\big)$, and $(x_k, w_k^k + \beta) \to (x, w)$, which implies that $(x, w) \in \text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big)$. If instead $(\check{\text{cl}}\, f)(x) = -\infty$, then there exists $(x, w') \in \text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big)$ such that $w' \leq w$. Then there exists $\{(x_k, w_k')\} \subset \text{conv}\big(\text{epi}(f)\big)$ such that $(x_k, w_k') \to (x, w')$. Denote $w - w'$ as β' and $\{(x_k, w_k' + \beta')\} \subset \text{conv}\big(\text{epi}(f)\big)$ and $(x_k, w_k' + \beta') \to (x, w)$, which concludes the proof. ☐

P. 38 The convex closure of a function $f : X \to [-\infty, \infty]$ is defined as

$$(\check{\text{cl}}\, f)(x) = \inf \big\{ w \,\big|\, (x, w) \in \text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big) \big\},\ \forall x \in \mathbb{R}^n.$$

Lemma (4, P. 38)**.** *Given a function* $f : X \to [-\infty, \infty]$*, its convex closure* $\check{\text{cl}}\, f$ *is the closure of function* $F : \mathbb{R}^n \to [-\infty, \infty]$*, defined as*

$$F(x) = \inf \big\{ w \,\big|\, (x, w) \in \text{conv}\big(\text{epi}(f)\big) \big\},\ \forall x \in \mathbb{R}^n. \tag{19}$$

Proof. In view of [Lemma 1, P. 38], we show that $\text{epi}(\check{\text{cl}}\, f) = \text{cl}\big(\text{epi}(F)\big)$. In view of [Lemma 3, P. 38], we in turn need to show that $\text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big) = \text{cl}\big(\text{epi}(F)\big)$.

For every $(x, w) \in \text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big)$, $\exists \{(x_k, w_k)\} \subset \text{conv}\big(\text{epi}(f)\big)$ such that $(x_k, w_k) \to (x, w)$. Since $\text{conv}\big(\text{epi}(f)\big) \subset \text{epi}(F)$, so we have $(x, w) \in \text{cl}\big(\text{epi}(F)\big)$.

To show the reverse, assume $(x, w) \in \text{cl}\big(\text{epi}(F)\big)$. Then $\exists \{(x_k, w_k)\} \subset \text{epi}(F)$ such that $(x_k, w_k) \to (x, w)$. For every k, $F(x_k) \leq w_k$. By the definition of F, if $F(x_k) > -\infty$, then exists $(x_k, z_k) \in \text{conv}\big(\text{epi}(f)\big)$ such that $z_k \in [F(x_k), F(x_k) + 1/k]$. By [Lemma 2, P. 38], we then have $(x_k, w_k + 1/k) \in \text{conv}\big(\text{epi}(f)\big)$. If $F(x_k) = -\infty$, then there exists $(x_k, z_k) \in \text{conv}\big(\text{epi}(f)\big)$ such that $z_k \leq w_k$, and by [Lemma 2, P. 38], we then have $(x_k, w_k + 1/k) \in \text{conv}\big(\text{epi}(f)\big)$. By above arguments, we see that $\{(x_k, w_k + 1/k)\} \subset \text{conv}\big(\text{epi}(f)\big)$ and $(x_k, w_k + 1/k) \to (x, w)$. Therefore $(x, w) \in \text{cl}\big(\text{conv}\big(\text{epi}(f)\big)\big)$, which concludes the proof. ☐

Notice that conv(epi(f)) may not be an epigraph. To see this, consider function $f : (-1, 1) \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x + 1, & x \in (-1, 0] \\ -x + 1, & x \in (0, 1) \end{cases}$$

Then one can see $\text{conv}(\text{epi}(f)) = \{(x, y) \,|\, x \in (-1, 1),\, y > 0\}$, which is not an epigraph. To see another example, consider $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 1/(1+x^2)$. Then we have $\text{conv}(\text{epi}(f)) = \{(x, y) \,|\, y > 0\}$. The point of emphasis here for the second example is that the function f is now closed, but the convex hull of its epigraph is still not an epigraph itself.

P. 38 Refer to [COT Prop. 1.3.6 (b), P. 29] and [Lemma, P. 29].

P. 39 Such sequence exists and can be constructed in the following way. Denote $R = \{y \,|\, y = (\text{cl}\, f)(x),\, x \in \mathbb{R}^n\}$. Then $f^* = \inf R$. If $f^* > -\infty$, then $\forall k$, there exists $\bar{x}_k \in \mathbb{R}^n$ and $y_k \in R$ such that $y_k < f^* + 1/k$ and $y_k = (\text{cl}\, f)(\bar{x}_k)$. In addition, $y_k \geq \inf R > -\infty$, so $(\bar{x}_k, y_k) \in \text{cl}\big(\text{epi}(f)\big)$. Therefore, setting $\bar{w}_k = y_k$ will do. Otherwise, $f^* = -\infty$. Then for all $-k - 1$, there exists $\bar{x}_k \in \mathbb{R}^n$ and $y_k \in R$ such that $y_k < -k - 1$ and $y_k = (\text{cl}\, f)(\bar{x}_k)$. Then $(\bar{x}_k, -k) \in \text{cl}\big(\text{epi}(f)\big)$ as $-k > -k - 1 > y_k$. Then setting $\bar{w}_k = -k$ will do. Note that the set R could be of the form $\{-\infty\}$, in which case $\inf\{-\infty\} = -\infty$.

P. 40 To see this, note that $f \geq \text{cl}\, f$. Therefore, $\text{cl}\, f$ being proper implies that $f(x) > -\infty\ \forall x$. In addition, since $\text{cl}\, f$ is proper, we have $\text{epi}(\text{cl}\, f) = \text{cl}\big(\text{epi}(f)\big)$ is nonempty and convex. Therefore, $\text{dom}(\text{cl}\, f)$, which is the projection of $\text{epi}(\text{cl}\, f)$, is nonempty and convex. Therefore, $\text{ri}\big(\text{dom}(\text{cl}\, f)\big)$ is convex and nonempty. By above arguments in the proof, we have $f(x) = (\text{cl}\, f)(x) \in \mathbb{R}$ for all $x \in \text{ri}\big(\text{dom}(\text{cl}\, f)\big)$. Therefore, f is proper.

P. 41 To see that g is convex and closed, we fill in some details. Define set $\bar{G}$ as $\bar{G} = \{(z, t) \,|\, z = y + \alpha(x - y),\, \alpha \in [0, 1],\, t \in \mathbb{R}\}$. Then define as G the set $\text{epi}(\text{cl}\, f) \cap \bar{G}$. Since $\text{epi}(\text{cl}\, f)$, $\bar{G}$ are both closed and convex, then by [COT Prop. 1.1.1 (a), P. 2] and [COT Prop. A.2.4 (b), P. 235], we have that G is closed and convex. Define affine function $h : \mathbb{R}^2 \to \mathbb{R}^{n+1}$ as $h(\alpha, \omega) = A[\alpha, \omega]' + b$ where A and b are

$$A = \begin{bmatrix} x - y & 0 \\ 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} y \\ 0 \end{bmatrix}.$$

Then one can see that $\text{epi}(g) = h^{-1}(G)$. Due to [COT Prop. A.2.6 (c), P. 237] and [COT Prop. 1.1.1 (e), P. 3], we see that $\text{epi}(g)$ is closed and convex.

P. 41 For an example, consider function $f : \mathbb{R}^2 \to [-\infty, \infty]$ as

$$f(x, y) = \begin{cases} 0, & (x, y) = (0, 0), \\ -\infty, & x > 0,\ y = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Then one can see that the function is convex and improper, taking $-\infty$ at all points in $\text{ri}\big(\text{dom}(f)\big)$.

P. 42 The result of [COT Prop. 1.3.8, P. 32] can be extended to any finitely many nonempty convex sets, that is, for C_i nonempty convex sets $i = 1, \ldots, m$, if $\cap_{i=1}^m \mathrm{ri}(C_i) \neq \emptyset$, then $\cap_{i=1}^m \mathrm{ri}(C_i) = \mathrm{ri}(\cap_{i=1}^m C_i)$. We take $m = 3$ as an example. Given $\cap_{i=1}^3 \mathrm{ri}(C_i) \neq \emptyset$, then $\cap_{i=1}^2 \mathrm{ri}(C_i) \neq \emptyset$, and we have $\cap_{i=1}^2 \mathrm{ri}(C_i) = \mathrm{ri}(\cap_{i=1}^2 C_i)$. Then it holds that $\cap_{i=1}^3 \mathrm{ri}(C_i) = \cap_{i=1}^2 \mathrm{ri}(C_i) \cap \mathrm{ri}(C_3) = \mathrm{ri}(C_1 \cap C_2) \cap \mathrm{ri}(C_3)$. Since $\cap_{i=1}^3 \mathrm{ri}(C_i) \neq \emptyset$, then $\mathrm{ri}(C_1 \cap C_2) \cap \mathrm{ri}(C_3) \neq \emptyset$. Therefore, $\mathrm{ri}(C_1 \cap C_2) \cap \mathrm{ri}(C_3) = \mathrm{ri}(\cap_{i=1}^3 C_i)$.

P. 43 For a nonempty convex set C, its recession cone R_C is always nonempty and containing origin.

P. 45 More precisely, due to the fact that $\|z_k\| \to \infty$.

P. 45 [COTe1] Ex 1.36 (a), P. 43. Alternatively, this can be proved as follows. Since by [COT Prop. 1.3.5 (b), P. 28] we have $\mathrm{ri}\big(\mathrm{cl}(C)\big) = \mathrm{ri}(C)$, and by [COT Prop. 1.4.2 (b), P. 45] that $R_C = R_{\mathrm{ri}(C)}$ when C is closed, then the result follows.

P. 46 Such a limit point d exists since the sequence $\{d_k\}$ has constant norm 1 and therefore is bounded. In addition, this also implies that the limit point d has norm 1 therefore is nonzero. To see that it is without loss of generality to assume $\|z_k - x\|$ is monotonically increasing, we first note that given any unbounded $\{z_k\}$, there is a subsequence that is monotonically increasing and unbounded. To see this, we first note that given 1, there is some k^1 such that $z_{k^1} > 1$. Then given i and z_{k^i}, there exists $k^{i+1} > k^i$ such that $z_{k^{i+1}} > \max\{i+1, z_{k^i}\}$ due to $\{z_k\}$ being unbounded. Denote the index set $\{k^i\}$ as $\mathcal{K}$. Then for every $\|z_i - x\|$ with $i \in \mathcal{K}$, there exists K such that for all $k \geq K$ and $k \in \mathcal{K}$, $\|z_k\| \geq \|z_i - x\| + \|x\|$, which implies $\|z_j - x\| \geq \|z_i - x\|$ for some $j > i$. Therefore, there exists a subsequence of $\{z_k\}$ where $\|z_k - x\|$ is monotonically increasing.

P. 46 Note that with C_i being convex, and $\cap_{i\in I} C_i \neq \emptyset$, it always holds that $\cap_{i\in I} R_{C_i} \subset R_{\cap_{i\in I} C_i}$, without requiring C_i being closed. We may give a symbolic elaboration. We have per definition of set intersection,

$$d \in \cap_{i\in I} R_{C_i} \iff \forall i (i \in I \implies d \in R_{C_i}), \tag{20}$$

$$x \in \cap_{i\in I} C_i \iff \forall i (i \in I \implies x \in C_i). \tag{21}$$

Then we have $\forall d$,

$$\begin{aligned} & d \in \cap_{i\in I} R_{C_i} \\ \iff & \forall i (i \in I \implies d \in R_{C_i}) \wedge \forall x \big(x \in \cap_{i\in I} C_i \iff \forall i (i \in I \implies x \in C_i) \big) \\ \implies & \forall x \big(x \in \cap_{i\in I} C_i \implies \forall i (i \in I \implies x \in C_i \wedge d \in R_{C_i}) \big), \end{aligned} \tag{22}$$

where Eq. (22) can be interpreted as applying the inference rule of the right-hand side of Eq. (20) to the right-hand side of Eq. (21). Now we focus on the condition $\forall i\big(i \in I \implies d \in R_{C_i} \wedge x \in C_i\big)$, and we have

$$
\begin{aligned}
&\forall i\big(i \in I \implies d \in R_{C_i} \wedge x \in C_i\big) \\
\implies &\forall i\big(i \in I \implies d \in R_{C_i} \wedge x \in C_i \implies \forall \alpha(\alpha \geq 0 \implies x + \alpha d \in C_i)\big) && (23) \\
\implies &\forall i\big(i \in I \implies \forall \alpha(\alpha \geq 0 \implies x + \alpha d \in C_i)\big) \\
\implies &\forall \alpha\big(\alpha \geq 0 \implies \forall i(i \in I \implies x + \alpha d \in C_i)\big) && (24) \\
\implies &\forall \alpha(\alpha \geq 0 \implies x + \alpha d \in \cap_{i\in I} C_i) && (25)
\end{aligned}
$$

Eq. (23) is per the definition of recession cone, Eq. (24) is due to the following facts

$$
\begin{aligned}
&\forall x \in X \forall y \in Y P(x,y) \iff \forall y \in Y \forall x \in X P(x,y), && (26) \\
&\forall x \in X \forall y \in Y P(x,y) \iff \forall x\big(x \in X \implies \forall y(y \in Y \implies P(x,y))\big), && (27) \\
&\forall x\big(x \in X \implies \forall y(y \in Y \implies P(x,y))\big) \iff \forall x \forall y\big(x \in X \wedge y \in Y \implies P(x,y)\big), && (28)
\end{aligned}
$$

where Eq. (28) can be directly proven, and Eq. (25) is per the definition of $\cap_{i\in I} C_i$. Collect above results, we have $\forall d$,

$$
\begin{aligned}
&d \in \cap_{i\in I} R_{C_i} \\
\implies &\forall x\big(x \in \cap_{i\in I} C_i \implies \forall \alpha(\alpha \geq 0 \implies x + \alpha d \in \cap_{i\in I} C_i)\big) \\
\iff &\forall x \forall \alpha(x \in \cap_{i\in I} C_i \wedge \alpha \geq 0 \implies x + \alpha d \in \cap_{i\in I} C_i) && (29)
\end{aligned}
$$

where Eq. (29) is due to Eq. (28). In fact, in Eq. (23) we have also used Eq. (28). That is, if orthodox definition of $d \in R_C$ is given as $\forall x \forall \alpha(x \in C \wedge \alpha \geq 0 \implies x + \alpha d \in C)$, then we have

$$
\begin{aligned}
d \in R_{C_i} \wedge x \in C_i &\iff x \in C_i \wedge \forall x' \forall \alpha(x' \in C_i \wedge \alpha \geq 0 \implies x' + \alpha d \in C_i) \\
&\iff x \in C_i \wedge \forall x'\big(x' \in C_i \implies \forall \alpha(\alpha \geq 0 \implies x' + \alpha d \in C_i)\big) \\
&\implies \forall \alpha(\alpha \geq 0 \implies x + \alpha d \in C_i).
\end{aligned}
$$

In fact, if the direction of recession and recession cone are also defined for any nonempty set, then given any group of sets C_i such that $\cap_{i\in I} C_i \neq \emptyset$, it would still hold that $\cap_{i\in I} R_{C_i} \subset R_{\cap_{i\in I} C_i}$, without requiring C_i being convex or closed, which can be seen from the proof given above. Therefore, in what follows, for any statements that we prove, if the results still hold without requiring convexity, given that the direction of recession and recession cone are defined for any nonempty set (instead of confined to nonempty convex set), we will write the statements as '... (convex) ...', namely putting 'convex' in parentheses, indicating that the term 'convex' is in the statements solely because that the direction of recession and recession cone are defined only for nonempty convex sets in this book.

P. 49 For a symmetric positive semidefinite $n \times n$ matrix Q, it holds that

$$d'Qd = 0 \iff Md = 0 \iff Qd = 0,$$

where $Q = M'M$. The first $\iff$ is obvious. For the second one, given $Md = 0$, we have $Qd = M'0 = 0$, and given $Qd = 0$, we have $d'Qd = 0$, which implies $Md = 0$.

P. 50 It is clear from the proof that $C \cap S^{\perp}$ is always nonempty.

P. 51 We fill in some details here. Per definition of S, we have

$$\forall x \forall v \big((x, v) \in S \iff x \in V_\gamma \wedge v = \gamma \big). \tag{30}$$

By the definition of recession cone, we have $\forall d \forall w$, $(d, w) \in R_S$ if and only if

$$\begin{aligned}
&\forall x \forall v \forall \alpha \big((x, v) \in S \wedge \alpha \geq 0 \implies (x, v) + \alpha(d, w) \in S \big) \\
\iff &\forall x \forall v \forall \alpha \big(x \in V_\gamma \wedge v = \gamma \wedge \alpha \geq 0 \implies x + \alpha d \in V_\gamma \wedge v + \alpha w = \gamma \big) \\
\iff &\forall x \forall v \forall \alpha \big(x \in V_\gamma \wedge v = \gamma \wedge \alpha \geq 0 \implies x + \alpha d \in V_\gamma \wedge \alpha w = 0 \wedge v = \gamma \big) \\
\iff &\forall x \forall \alpha \big(x \in V_\gamma \wedge \alpha \geq 0 \implies x + \alpha d \in V_\gamma \wedge \alpha w = 0 \big) \\
\iff &\forall x \forall \alpha \big(x \in V_\gamma \wedge \alpha \geq 0 \implies x + \alpha d \in V_\gamma \big) \wedge \forall \alpha (\alpha \geq 0 \implies \alpha w = 0) \\
\iff &d \in R_{V_\gamma} \wedge w = 0,
\end{aligned}$$

where the first step is due to Eq. (30), and the last step is due to the definition of R_{V_γ} and property of 0. Then collect above results, we have

$$\forall d \forall w \big(d \in R_{V_\gamma} \wedge w = 0 \iff (d, w) \in R_S \big). \tag{31}$$

On the other hand, we also have

$$\forall d \forall w \big((d, w) \in R_S \iff (d, 0) \in R_{\mathrm{epi}(f)} \wedge w = 0 \big). \tag{32}$$

Then combining above two equations, we have

$$\begin{aligned}
&\forall d \forall w \big(d \in R_{V_\gamma} \wedge w = 0 \iff (d, 0) \in R_{\mathrm{epi}(f)} \wedge w = 0 \big) \\
\iff &\forall d \big(d \in R_{V_\gamma} \iff (d, 0) \in R_{\mathrm{epi}(f)} \big),
\end{aligned}$$

which is the desired result.

P. 51 In view of [COT Prop. 1.1.2, P. 10], V_γ is always closed.

P. 52

Lemma (1, P. 52)**.** *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a closed proper convex function, and $x \in \mathrm{dom}(f)$. If $d \in R_f$ and $\beta \geq \alpha \geq 0$, then $f(x + \alpha d) \geq f(x + \beta d)$.*

Proof. Denote $\gamma = f(x)$. Then $x + \alpha d$, $x + \beta d \in V_\gamma$ due to the definition of d. Therefore, $f(x + \alpha d)$ and $f(x + \beta d)$ are both finite. Then consider V_κ with $\kappa = f(x + \alpha d)$. Then we see that $x + \beta d = x + \alpha d + (\beta - \alpha) d \in V_\kappa$, which implies $f(x + \beta d) \leq \kappa$. □

P. 52

Lemma (2, P. 52)**.** *Let* $f : \mathbb{R}^n \to (-\infty, \infty]$ *be a closed proper convex function, and* $x \in V_\gamma$*, where* V_γ *is a nonempty level set of* f*. If* $x + \alpha d \notin V_\gamma$ *for some* d *and* $\alpha > 0$*, then* $x + \beta d \notin V_\gamma$ *for all* $\beta \geq \alpha$*.*

Proof. Since f is convex, then V_γ is convex. Assume above is false, namely, there exists $\beta \geq \alpha$, such that $x + \beta d \in V_\gamma$. Then in view of V_γ being convex, this implies that $x + \alpha d \in V_\gamma$, which is a contradiction. □

P. 52

Lemma (3, P. 52)**.** *Let* $f : \mathbb{R}^n \to (-\infty, \infty]$ *be a closed proper convex function, and* $x \in V_\gamma$*, where* V_γ *is a nonempty level set of* f*. If* $d \notin R_f$*, then there exists some* $\alpha > 0$*, such that for all* β*,* κ *with* $\alpha \leq \beta < \kappa$*,*

i) if $f(x + \beta d) = \infty$*, then* $f(x + \kappa d) = \infty$*;*

ii) if $f(x + \beta d) < \infty$*, then* $f(x + \beta d) < f(x + \kappa d)$*.*

In addition, it holds that $\sup\{f(x + \delta d) \,|\, \delta \geq 0\} = \infty$*.*

Proof. Since $x \in V_\gamma$ and $d \notin R_f$, then there exists some $\alpha > 0$ such that $x + \alpha d \notin V_\gamma$. Then it holds that $f(x + \alpha d) > \gamma \geq f(x)$. In view of [Lemma 2, P. 52], it holds that for all β, $\kappa \in [\alpha, \infty)$, we have $f(x + \beta d)$, $f(x + \kappa d) \in (\gamma, \infty]$. If $f(x + \beta d) = \infty$, then in view of convexity of f, we have $f(x + \kappa d) = \infty$. If $f(x + \beta d) < \infty$, then we have

$$x + \beta d = \frac{\beta}{\kappa}(x + \kappa d) + \frac{\kappa - \beta}{\kappa} x.$$

Then by convexity of f, we have

$$f(x+\beta d) \leq \frac{\beta}{\kappa} f(x+\kappa d) + \frac{\kappa - \beta}{\kappa} f(x) \iff f(x+\beta d) - \frac{\kappa - \beta}{\kappa} f(x) \leq \frac{\beta}{\kappa} f(x+\kappa d).$$

Since $f(x + \beta d) > \gamma \geq f(x)$ and $f(x + \beta d) < \infty$, we have

$$f(x + \beta d) - \frac{\kappa - \beta}{\kappa} f(x + \beta d) < \frac{\beta}{\kappa} f(x + \kappa d) \iff f(x + \beta d) < f(x + \kappa d).$$

Regarding the unboundedness of $\sup\{f(x + \delta d) \,|\, \delta \geq 0\}$, note that for all $k \in \mathbb{N}$, we have $x \in V_{\gamma+k}$. Since $d \notin R_f$, then there exists δ_k such that $x + \delta_k d \notin V_{\gamma+k}$, which implies $f(x + \delta_k) > \gamma + k$. Therefore $\sup\{f(x + \delta d) \,|\, \delta \geq 0\} = \infty$. □

P. 54

Lemma (1, P. 54)**.** *Let* $A_1 \in \mathbb{R}^m$*,* $A_2 \in \mathbb{R}^n$ *be nonempty sets. If* $A_1 \times A_2$ *is closed, then both* A_1 *and* A_2 *are closed.*

Proof. We take A_2 as an example. We apply $\|\cdot\|_\infty$ for all three spaces $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^{m+n}$. For any $\{a_2^k\} \subset A_2$ that is convergent in $\mathbb{R}^n$, we see that for every $a_1 \in A_1$, the sequence $\{(a_1, a_2^k)\} \subset A_1 \times A_2$ is convergent. Since $A_1 \times A_2$ is closed, there exists $(\bar{a}_1, \bar{a}_2) \in A_1 \times A_2$ such that $\lim_{k\to\infty} \|(a_1, a_2^k) - (\bar{a}_1, \bar{a}_2)\|_\infty = 0$. Since $\bar{a}_2 \in A_2$ due to $(\bar{a}_1, \bar{a}_2) \in A_1 \times A_2$ and $\|a_2^k - \bar{a}_2\|_\infty \leq \|(a_1, a_2^k) - (\bar{a}_1, \bar{a}_2)\|_\infty$, we see that A_2 is closed. □

Lemma (2, P. 54)**.** *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a closed proper convex function. The recession cone of its epigragh* $\mathrm{epi}(f)$*, denoted as $R_{\mathrm{epi}(f)}$, is the epigraph of a closed proper convex function.*

Proof. Since $f : \mathbb{R}^n \to (-\infty, \infty]$ is a closed proper convex function, then $\mathrm{epi}(f)$ is a nonempty closed convex set. Then by [COT Prop. 1.4.1 (a), P. 43], $R_{\mathrm{epi}(f)}$ is a nonempty closed convex set. For every $d \in \mathbb{R}^n$, denote as M_d the set $\{(d, v) \,|\, v \in \mathbb{R}\}$. Note that for every d, M_d is closed, and therefore so is $M_d \cap R_{\mathrm{epi}(f)}$. In addition, let $T : \mathbb{R}^{n+1} \to \mathbb{R}$ be the projection mapping $T(d, v) = v$. Then for every d, one can prove via definition that

$$M_d \cap R_{\mathrm{epi}(f)} = \{d\} \times T\big(M_d \cap R_{\mathrm{epi}(f)}\big).$$

Then for every d such that $M_d \cap R_{\mathrm{epi}(f)} \neq \emptyset$, due to [Lemma 1, P. 54], we see that $T\big(M_d \cap R_{\mathrm{epi}(f)}\big)$ is closed. Then what left to show is for every d such that $M_d \cap R_{\mathrm{epi}(f)} \neq \emptyset$, the set $T\big(M_d \cap R_{\mathrm{epi}(f)}\big)$ shall be a closed interval that is bounded below and unbounded above. For $d = 0$, it is obvious that $T\big(M_0 \cap R_{\mathrm{epi}(f)}\big) = [0, \infty)$ since f is proper. For every $d \neq 0$ such that $M_d \cap R_{\mathrm{epi}(f)} \neq \emptyset$, there exists $(d, v) \in R_{\mathrm{epi}(f)}$. Then for every $x \in \mathrm{dom}(f)$, we have $-\infty < f(x + \alpha d) \leq f(x) + \alpha v < \infty$ for all $\alpha \geq 0$, since $\big(x, f(x)\big) \in \mathrm{epi}(f)$. In addition, $\forall v' \in T\big(M_d \cap R_{\mathrm{epi}(f)}\big)$, it holds for every fixed $\alpha \geq 0$ that $-\infty < f(x + \alpha d) \leq f(x) + \alpha v' < \infty$, which means $\underline{v}_d = \inf T\big(M_d \cap R_{\mathrm{epi}(f)}\big) \in \mathbb{R}$. Since $T\big(M_d \cap R_{\mathrm{epi}(f)}\big)$ is closed, then $\underline{v}_d \in T\big(M_d \cap R_{\mathrm{epi}(f)}\big)$. It is obvious that $\forall v' \geq \underline{v}_d$, $v' \in T\big(M_d \cap R_{\mathrm{epi}(f)}\big)$ due to the definition of epigraph, which concludes the proof. □

P. 55 Per definition, we have $\forall d \in \mathbb{R}^n$,

$$r_f(d) = \inf\{v \in \mathbb{R} \,|\, (d, v) \in R_{\mathrm{epi}(f)}\}.$$

Since f is proper, we have $\forall x \in \mathrm{dom}(f)$, $(x, f(x)) \in \mathrm{epi}(f)$. In addition, since f is proper, closed and convex, we have $\forall x \in \mathrm{dom}(f)$, $(x, f(x)) \in \mathrm{epi}(f)$, which implies $\forall d \in \mathbb{R}^n$ and $\forall v \in \mathbb{R}$,

$$\begin{aligned}
(d, v) \in R_{\mathrm{epi}(f)} &\iff \forall \alpha \geq 0 \big((x + \alpha d, f(x) + \alpha v) \in R_{\mathrm{epi}(f)}\big) && (33)\\
&\iff \forall \alpha > 0 \big((x + \alpha d, f(x) + \alpha v) \in R_{\mathrm{epi}(f)}\big) && (34)\\
&\iff \sup_{\alpha > 0} \frac{f(x + \alpha d) - f(x)}{\alpha} \leq v, && (35)
\end{aligned}$$

where (33) is due to the [COT Prop. 1.4.1 (b), P. 43] and the condition $(x, f(x)) \in \text{epi}(f)$, (34) is due to the condition $(x, f(x)) \in \text{epi}(f)$, and (35) is due to the definition of supremum and $f(x) \in \mathbb{R}$ so that $f(x + \alpha d) - f(x)$ is properly defined. Therefore, we have $\forall x \in \text{dom}(f)$, $\forall d \in \mathbb{R}^n$,

$$\{v \in \mathbb{R} \,|\, (d, v) \in R_{\text{epi}(f)}\} = \Big\{v \in \mathbb{R} \,\Big|\, \sup_{\alpha > 0} \frac{f(x + \alpha d) - f(x)}{\alpha} \le v\Big\}.$$

Therefore, we have $\forall x \in \text{dom}(f)$, $\forall d \in \mathbb{R}^n$,

$$r_f(d) = \inf \Big\{v \in \mathbb{R} \,\Big|\, \sup_{\alpha > 0} \frac{f(x + \alpha d) - f(x)}{\alpha} \le v\Big\},$$

which in turn implies that

$$r_f(d) = \sup_{\alpha > 0} \frac{f(x + \alpha d) - f(x)}{\alpha}, \tag{36}$$

regardless of the choice of $x \in \text{dom}(f)$. Another interesting point to note is that for d such that $r_f(d) = \inf \emptyset = \infty$, we have $\frac{f(x+\alpha d) - f(x)}{\alpha} = \infty$ for all $x \in \text{dom}(f)$.

P. 56 We fill in some details here. For $0 < \alpha \le \beta$, we have $x + \alpha d = x + (\alpha/\beta)(x + \beta d - x) = (1 - \alpha/\beta)x + (\alpha/\beta)(x + \beta d)$ for all x. Then due to f being proper convex, we have

$$\frac{f(x + \alpha d) - f(x)}{\alpha} \le \frac{(1 - \alpha/\beta) f(x) + (\alpha/\beta) f(x + \beta d) - f(x)}{\alpha} = \frac{f(x + \beta d) - f(x)}{\beta}.$$

To see that $\lim_{\alpha \to \infty} \frac{f(x+\alpha d) - f(x)}{\alpha}$ is properly defined, we first note that by above arguments, we have $r_f(d) = \sup_{\alpha > 0} \frac{f(x+\alpha d) - f(x)}{\alpha} > -\infty$ due to f being proper and [Lemma 2, P. 54]. We take $r_f(d) \in \mathbb{R}$ as an example and the case $r_f(d) = \infty$ is entirely similar. Define $g_{x,d} : (0, \infty) \to \mathbb{R}^*$ as $g_{x,d}(\alpha) = \frac{f(x+\alpha d) - f(x)}{\alpha}$ where $x \in \text{dom}(f)$. Then one can see that $g_{x,d}(\alpha) > -\infty$ for all $\alpha > 0$, and by above arguments it is nondecreasing. We then show that for every sequence $\{\alpha_k\}$ such that $\alpha_k > 0$ and $\alpha_k \to \infty$, $\lim_{k \to \infty} g_{x,d}(\alpha_k) = r_f(d)$. To see this, due to the definition of supremum and $r_f(d) = \sup_{\alpha > 0} \frac{f(x+\alpha d) - f(x)}{\alpha} \in \mathbb{R}$, for every $\varepsilon > 0$, there exists $\beta > 0$ such that $g_{x,d}(\beta) > r_f(d) - \varepsilon$. Since $\alpha_k \to \infty$ and $g_{x,d}$ is nondreasing, then there exists some K such that $\alpha_k > \beta$ for all $k > K$ and therefore $g_{x,d}(\alpha_k) > r_f(d) - \varepsilon$. On the other hand, $g_{x,d}(\alpha_k) \le r_f(d) = \sup_{\alpha > 0} g_{x,d}(\alpha)$. So per definition, $\lim_{k \to \infty} g_{x,d}(\alpha_k) = r_f(d)$. For the case where $r_f(d) = \infty$, similar arguments can be applied. Therefore, for every sequence $\{\alpha_k\}$ such that $\alpha_k > 0$ and $\alpha_k \to \infty$, $\lim_{k \to \infty} g_{x,d}(\alpha_k) = r_f(d)$, which means $\lim_{\alpha \to \infty} g_{x,d}(\alpha) = r_f(d)$.

P. 56 To see that $\lim_{\alpha \to \infty} \nabla f(x + \alpha d)' d$ is well-defined, we define $g_{x,d} : (0, \infty) \to \mathbb{R}^*$ as $g_{x,d}(\alpha) = \frac{f(x+\alpha d) - f(x)}{\alpha}$ where $x \in \text{dom}(f)$. Note that for every sequence $\{\alpha_k\}$ where $\alpha_k > 0$ and $\alpha_k \to \infty$, it holds that

$$g_{x,d}(\alpha_k) \le \nabla f(x + \alpha_k d)' d,$$

where $\nabla f(x+\alpha_k d)'d \in \mathbb{R}$. Take limit inferior on both sides and we have

$$r_f(d) \le \liminf_{k\to\infty} \nabla f(x+\alpha_k d)'d. \tag{37}$$

Similarly, due to the following holds

$$\nabla f(x+\alpha_k d)'d \le r_f(d),$$

we also have

$$\limsup_{k\to\infty} \nabla f(x+\alpha_k d)'d \le r_f(d). \tag{38}$$

Combining Eqs. (37) and (38), we see for every $\{\alpha_k\}$ where $\alpha_k > 0$ and $\alpha_k \to \infty$, $\lim_{k\to\infty} \nabla f(x+\alpha_k d)'d = r_f(d)$. Therefore, $\lim_{\alpha\to\infty} \nabla f(x+\alpha d)'d = r_f(d)$.

P. 57 The first and the third equalities are due to [COT Prop. 1.4.7, P. 55]. To see the second equality hold, for $x \in \mathrm{dom}(f_1+f_2)$ and all $d \in \mathbb{R}^n$, we denote $g^i_{x,d} : (0,\infty) \to \mathbb{R}^*$ as $g^i_{x,d}(\alpha) = \frac{f_i(x+\alpha d)-f_i(x)}{\alpha}$ with $i = 1, 2$. Then from [COT Note P. 56], due to f_i being proper, closed, and convex, for any $\{\alpha_k\}$ such that $\alpha_k > 0$ and $\alpha_k \to \infty$, $\{g^i_{x,d}(\alpha_k)\} \subset (-\infty,\infty]$, nondecreasing, with limit in $(-\infty,\infty]$. Similarly, due to f_1+f_2 being proper, closed, and convex, $\{g^1_{x,d}(\alpha_k)+g^2_{x,d}(\alpha_k)\} \subset (-\infty,\infty]$ is nondecreasing, and has limit in $(-\infty,\infty]$. Then due to [Lemma 1, P. 13], for the chosen $\{\alpha_k\}$, we have

$$\lim_{k\to\infty} g^1_{x,d}(\alpha_k) + \lim_{k\to\infty} g^2_{x,d}(\alpha_k) = \lim_{k\to\infty} \big(g^1_{x,d}(\alpha_k)+g^2_{x,d}(\alpha_k)\big).$$

Since per [COT Note P. 56], we have $\lim_{\alpha\to\infty} g^i_{x,d}(\alpha) = \lim_{k\to\infty} g^i_{x,d}(\alpha_k)$, $i = 1, 2$, and $\lim_{\alpha\to\infty}\big(g^1_{x,d}(\alpha)+g^2_{x,d}(\alpha)\big) = \lim_{k\to\infty}\big(g^1_{x,d}(\alpha_k)+g^2_{x,d}(\alpha_k)\big)$, the second equality follows.

P. 58

Lemma (P. 58)**.** *Let $\{C_k\}$ be a nested sequence of nonempty closed sets in $\mathbb{R}^n$. Then $\cap_{k=1}^\infty C_k$ is empty if and only if every sequence $\{x_k\}$ that has the property $x_k \in C_k\ \forall k$ is unbounded.*

Proof. We first prove the if part. Assume $\cap_{k=1}^\infty C_k \ne \emptyset$ and $\bar{x} \in \cap_{k=1}^\infty C_k$. Then for the sequence $x_k \equiv \bar{x}$, we have $x_k \in C_k$ and the sequence is bounded.

For the only if part, we prove the result by contradiction. Given $\cap_{k=1}^\infty C_k = \emptyset$, if there exists $\{x_k\}$ such that it is bounded and $x_k \in C_k\ \forall k$, then we see that $\{x_i\}_{i=k}^\infty \subset C_k$ for all k. Denote as $\tilde{C}_k$ the closure of the set $\{x_i\}_{i=k}^\infty$, and $\tilde{C}_k$ is compact due to $\{x_i\}_{i=k}^\infty$ being bounded. Since C_k is closed and contains $\{x_i\}_{i=k}^\infty$, we have $\tilde{C}_k \subset C_k$ for all k. In addition, it is straightforward to see that $\{\tilde{C}_k\}$ is nested. Then by [COT Prop. A.2.4 (h), P. 235], we have $\cap_{k=1}^\infty \tilde{C}_k \ne \emptyset$. However, we also have $\cap_{k=1}^\infty \tilde{C}_k \subset \cap_{k=1}^\infty C_k = \emptyset$, which is a contradiction. This concludes the proof. $\square$

P. 59 Note that for any collection of (convex) sets $\{C_i\}_{i\in I}$, if $\cap_{i\in I} C_i \neq \emptyset$. then it holds that $\cap_{i\in I} R_{C_i} \subset R_{\cap_{i\in I} C_i}$. Refer to [COT Note P. 46] for more details. However, it is possible that $\{C_i\}_{i\in I} = \emptyset$ and $R_{\cap_{i\in I} C_i}$ is undefined, yet $\cap_{i\in I} R_{C_i}$ contains nonzero elements. One example is $C_i = [i, \infty)$ where $I = \mathbb{N}$. Then $1 \in \cap_{i\in I} R_{C_i}$ yet $\cap_{i\in I} C_i = \emptyset$. Note that for any collection of nonempty (convex) sets $\{C_i\}_{i\in I}$, $\cap_{i\in I} R_{C_i}$ is always nonempty since it contains 0.

P. 59

Lemma (1, P. 59)**.** *Let $\{C_k\}$ be a nested sequence of nonempty closed convex sets. Then given any sequence $\{x_k\}$ of nonzero vectors such that $x_k \in C_k$ for all k and $\|x_k\| \to \infty$, any limit point of $\{x_k/\|x_k\|\}$ is an element of $\cap_{k=0}^{\infty} R_{C_k}$.*

Proof. If there is no such sequence $\{x_k\}$, then the statement is vacuously true. Otherwise, since the sequence $\{x_k/\|x_k\|\}$ is bounded, it has convergent subsequence. Denote the index set of one such subsequence as $\mathcal{K}_1$ and its limit as d. Then we will show for every C_i, d is its direction of recession. First, we note that the limit of $\{x_k\}_{k\in\mathcal{K}_1}$ is also infinity. Then for any $x_i \in C_i$, there exists a subsequence of $\{x_k\}_{k\in\mathcal{K}_1}$ with index set $\mathcal{K}_2 \subset \mathcal{K}_1 \setminus \{0, \ldots, i\}$ such that $\|x_k\| > \|x_i\|$ for all $k \in \mathcal{K}_2$ and therefore $\|x_k - x_i\| \neq 0$, and $\{x_k\}_{k\in\mathcal{K}_2} \subset C_i$. In addition, due to construction, we have the limits of $\{x_k\}_{k\in\mathcal{K}_2}$ and $\{x_k/\|x_k\|\}_{k\in\mathcal{K}_2}$ to be ∞ and d respectively. Then we show that the limit of $\{(x_k - x_i)/\|x_k - x_i\|\}_{k\in\mathcal{K}_2}$ exists and is d. To see this, we write $(x_k - x_i)/\|x_k - x_i\|$ as

$$\frac{x_k - x_i}{\|x_k - x_i\|} = \frac{x_k}{\|x_k\|} \cdot \frac{\|x_k\|}{\|x_k - x_i\|} - \frac{x_i}{\|x_k - x_i\|}.$$

It is clear that the limits of $\{\|x_k\|/\|x_k - x_i\|\}_{k\in\mathcal{K}_2}$ and $\{x_i/\|x_k - x_i\|\}_{k\in\mathcal{K}_2}$ are 1 and 0 respectively. Therefore, the limit of $\{(x_k - x_i)/\|x_k - x_i\|\}_{k\in\mathcal{K}_2}$ exists and is d. Then we can apply the arguments for proving [COT Prop. 1.4.2 (a), P. 45-46] to show that d is a direction of recession of C_i, where it would also rely on the fact that C_i is closed and convex and [COT Prop. 1.4.1 (b), P. 43] can be applied. □

This result above proves that given a nested sequence $\{C_k\}$ of nonempty closed sets, if in addition the sets are convex, then any asymptotic directions of the nested sequence is also an element of $\cap_{k=0}^{\infty} R_{C_k}$. Refer to [Bet07] for the definition of asymptotic direction.

P. 59

Lemma (2, P. 59)**.** *Let $\{C_k\}$ be a nested sequence of nonempty closed convex sets. Then $\{C_k\}$ does not have asymptotic sequence if and only if every sequence $\{x_k\}$ that fulfills the condition $x_k \in C_k$ $\forall k$ is a bounded sequence.*

Proof. We first prove the if part. Since every sequence $\{x_k\}$ that fulfills the condition $x_k \in C_k$ is a bounded sequence, then no sequence fulfills $\|x_k\| \to \infty$.

Conversely, for the only if part, given that there exists an unbounded sequence $\{x_k\}$ such that $x_k \in C_k$, we apply [COT Note Lemma 1, P. 59], and we can construct an asymptotic sequence, which proves the only if part. Note that since [COT Note Lemma 1, P. 59] relies on the nested sets being convex, so the statement here also need the convexity condition. □

Lemma (3, P. 59)**.** *Let $\{C_k\}$ be a nested sequence of nonempty closed convex sets. Then $\{C_k\}$ does not have asymptotic sequence if and only if the set $\cap_{k=0}^{\infty} R_{C_k}$ is the singleton $\{0\}$.*

Proof. The if part of the statement is obvious per definition of asymptotic sequence. For the only if part, by [COT Note Lemma 2, P. 59], $\{C_k\}$ does not have asymptotic sequence if and only if every sequence $\{x_k\}$ that fulfills the condition $x_k \in C_k \ \forall k$ is a bounded sequence. We will show that this implies that there exists some index ℓ such that C_ℓ is bounded. Indeed, assume that C_k is unbounded for all k, then we can construct an unbounded sequence $\{x_k\}$ that fulfills the condition $x_k \in C_k \ \forall k$, which is a direct contradiction. Therefore, $\{C_k\}$ having no asymptotic sequence implies for some ℓ, C_ℓ is bounded. Then due to [COT Prop. 1.4.2 (a), P. 45], we have $\cap_{k=0}^{\infty} R_{C_k} = \{0\}$. □

The only if part of above proof relies on the convexity property of $\{C_k\}$ through the use of [COT Note Lemma 2, P. 59]. We will give an alternative proof which does not rely on the convexity property. The if part is neglected.

Proof. For the only if part, assume that $d \in \cap_{k=0}^{\infty} R_{C_k}$ and $d \neq 0$. Since C_k is nonempty, there is a sequence $\{z_k\}$ such that $z_k \in C_k$ and $z_k \neq 0$. Then we construct a sequence $\{x_k\}$ as follows. We set $x_0 = z_0$. Then we set $x_k = z_k + \alpha_k d$ such that $\|x_k\|/\|x_{k-1}\| > k$ and $\alpha_k \|d\|/\|z_k\| > k$ for some $\alpha_k > 0$. Due to d being a common direction of recession, we have $x_k \in C_k$ for all k. In addition, we have $\|x_k\| > 2^{k-1}\|x_0\|$. Therefore, $\|x_k\| \to \infty$. In addition, we have

$$\frac{x_k}{\|x_k\|} = \frac{z_k}{\|x_k\|} + \frac{\alpha_k d}{\|x_k\|}.$$

By triangular inequality, we have

$$\alpha_k \|d\| - \|z_k\| \leq \|x_k\| = \|z_k + \alpha_k d\| \leq \alpha_k \|d\| + \|z_k\|$$

Due to the construction of x_k, we have $\|z_k\| < \alpha_k \|d\|/k$. Therefore, we have

$$\frac{k-1}{k} \alpha_k \|d\| < \|x_k\| < \frac{k+1}{k} \alpha_k \|d\|.$$

Dividing above inequality with $\alpha_k \|d\|$ and taking limits gives $\|x_k\|/(\alpha_k \|d\|) \to 1$. Since we have

$$\frac{\|z_k\|}{\|x_k\|} = \frac{\|z_k\|}{\alpha_k \|d\|} \cdot \frac{\alpha_k \|d\|}{\|x_k\|} \to 0, \ \frac{\alpha_k d}{\|x_k\|} = \frac{\alpha_k d}{\alpha_k \|d\|} \cdot \frac{\alpha_k \|d\|}{\|x_k\|} \to \frac{d}{\|d\|},$$

Therefore, the sequence $\{x_k\}$ is asymptotic. □

P. 60

Lemma (1, P. 60)**.** *Let* $\{x_k\}$ *and* $\{y_k\}$ *be two positive sequences such that* $y_k \to \infty$ *and* $x_k/y_k \to a$ *where* $a \in (0, \infty)$*. Then* $x_k \to \infty$*.*

Proof. Since $x_k/y_k \to a$, then for some $\varepsilon > 0$ such that $a - \varepsilon > 0$, there exists some K_1 such that $x_k/y_k > a - \varepsilon$ for all $k > K_1$. In addition, since $y_k \to \infty$, then for every N, there exists K_2 such that $y_k > N/(a-\varepsilon)$. Therefore, $x_k > y_k(a - \varepsilon) > N$ for all $k > \max\{K_1, K_2\}$. Therefore, $x_k \to \infty$. □

Lemma (2, P. 60)**.** *Let* C^1*,* C^2*, ...,* C^r *be nonempty (convex) sets, and* R_{C^1}*, ...,* R_{C^r} *as their recession cones. Denote as* T *the Cartesian product* $C^1 \times C^2 \times \cdots \times C^r$*. Then it holds that*

$$R_T = R_{C^1} \times R_{C^2} \times \cdots \times R_{C^r}.$$

Proof. Denote as I the index set $\{1, 2, \ldots, r\}$. Per definition, $d = (d^1, \ldots, d^r) \in R_T$ if and only if $\forall t = (t^1, \ldots, t^r) \in T$, $\forall \alpha \geq 0$, it holds that $(t^1 + \alpha d^1, \ldots, t^r + \alpha d^r) \in T$. Per definition of T, the previous proposition is true if and only if $\forall i \in I$, $\forall c^i \in C^i$ and $\forall \alpha \geq 0$, $c^i + \alpha d^i \in C^i$, which in turn holds if and only if $\forall i \in I$, $d^i \in R_{C^i}$. □

We may give a more symbolic elaboration as follows.

Proof. $\forall d$, $d \in R_T$ if and only if (per definition of recession cone)

$$\begin{aligned}
&\forall t\big(t \in T \implies \forall \alpha \geq 0(t + \alpha d \in T)\big) \\
\iff &\forall t\Big(\big(\forall i(i \in I \implies t^i \in C^i)\big) \implies \big(\forall \alpha \geq 0 \forall i(i \in I \implies t^i + \alpha d^i \in C^i)\big)\Big) && (39) \\
\iff &\forall i\big(i \in I \implies \forall \alpha \geq 0 \forall c^i(c^i \in C^i \implies c^i + \alpha d^i \in C^i)\big) && (40) \\
\iff &\forall i(i \in I \implies d^i \in R_{C^i}) && (41) \\
\iff &d \in R_{C^1} \times R_{C^2} \times \cdots \times R_{C^r}, && (42)
\end{aligned}$$

where (39) is due to the definition of T being a Cartesian product, (40) can be proven, (41) is due to the definition of R_{C^i}, and (42) is due to the definition of $R_{C^1} \times R_{C^2} \times \cdots \times R_{C^r}$. □

Lemma (3, P. 60)**.** *Let* $\{C^1_k\}$*,* $\{C^2_k\}$*, ...,* $\{C^r_k\}$ *be retractive nested sequences of nonempty closed (convex) sets. Then* $\{T_k\}$ *is retractive nested sequence closed (convex) sets, where*

$$T_k = C^1_k \times C^2_k \times \cdots \times C^r_k.$$

Proof. Given any asymptotic sequence $\{x_k\}$ of $\{T_k\}$, we need to prove that it is retractive. Denote as I the index set $\{1, 2, \ldots, r\}$. Since $\{x_k\}$ is asymtotic, we have x_k is nonzero and

$$\|x_k\| \to \infty, \quad \frac{x_k}{\|x_k\|} \to \frac{d}{\|d\|},$$

where by [COT Note Lemma 2, P. 60], we have $d = (d^1, \ldots, d^r)$ and $d^i \in R_{C^i}$ for all $i \in I$. Denote as I_0 the set where $d^i = 0$ for $i \in I_0$, and $I_1 = I \setminus I_0$. Then we only need to focus on $\{x_k^i\}$ where $i \in I_1$. First, we note that for $i \in I_1$,

$$\frac{x_k^i}{\|x_k\|} \to \frac{d^i}{\|d\|} \neq 0 \implies \frac{\|x_k^i\|}{\|x_k\|} \to \frac{\|d^i\|}{\|d\|} \neq 0,$$

which in turn implies $\|x_k^i\| \to \infty$ in view of [COT Note Lemma 1, P. 60]. This means that there can only be finitely many zero terms in $\{x_k^i\}_{k=0}^{\infty}$. Therefore, without loss of generality, we assume the sequence is nonzero, and we have $\frac{\|x_k\|}{\|x_k^i\|} \to \frac{\|d\|}{\|d^i\|}$. Therefore, it holds that

$$\frac{x_k^i}{\|x_k^i\|} = \frac{x_k^i}{\|x_k\|} \cdot \frac{\|x_k\|}{\|x_k^i\|} \to \frac{d^i}{\|d\|} \cdot \frac{\|d\|}{\|d^i\|} = \frac{d^i}{\|d^i\|}.$$

Therefore, the sequence $\{x_k^i\}_{k=0}^{\infty}$ is asymptotic, therefore retractive. As a result, for $k > \bar{k}$, we have $x_k - d \in T$, where $\bar{k} = \max_{i \in I_1} \{\bar{k}_i\}_{i \in I_1}$, and $\bar{k}_i$ is the index for sequence $\{x_k^i\}_{k=0}^{\infty}$ to have the retractiveness property. □

P. 60

Lemma (4, P. 60)**.** *A closed half space C in $\mathbb{R}^n$ is retractive, where $C = \{x \mid a'x \leq b\}$, $a \in \mathbb{R}^n$, and b is a scalar.*

Proof. Given any asymptotic sequence $\{x_k\}$ where $x_k/\|x_k\| \to d/\|d\|$, we need to show that the sequence is retractive, namely, for some $\bar{k}$, it holds that

$$a'(x_k - d) \leq b, \ \forall k \geq \bar{k}. \tag{43}$$

First, we note that for the given asymptotic sequence $\{x_k\}$, since $a'x_k \leq b$, then $a'x_k/\|x_k\| \leq b/\|x_k\|$. Taking limits on both sides, we have $a'd/\|d\| \leq 0$, which implies $a'd \leq 0$. Now we assume that $\{x_k\}$ is not retractive. Then there exists a subsequence $\{x_k\}_{k \in \mathcal{K}}$ with indices $\mathcal{K}$ such that (43) does not hold, namely $a'x_k > a'd + b$ for all $k \in \mathcal{K}$. Therefore, we have for all $k \in \mathcal{K}$,

$$a'd + b < a'x_k \leq b \implies \frac{a'd + b}{\|x_k\|} < \frac{a'x_k}{\|x_k\|} \leq \frac{b}{\|x_k\|}. \tag{44}$$

Since we have

$$\lim_{k \in \mathcal{K}, k \to \infty} \|x_k\| = \infty, \quad \lim_{k \in \mathcal{K}, k \to \infty} \frac{x_k}{\|x_k\|} = \frac{d}{\|d\|},$$

then we take the limits of the right side inequalities of (44), we have $a'd = 0$. This contradicts the assumption that $a'x_k \le b$, and $a'x_k > a'd + b$ for all $k \in \mathcal{K}$. Therefore, the assumption is false and $\{x_k\}$ is retractive. $\square$

P. 62 Without loss of generality, we assume $\|x_k\| \neq 0 \; \forall k \in \overline{\mathcal{K}}$.

P. 62 Due to [COT Note Lemma 1, P. 59].

P. 65 Without loss of generality, we assume that $\|y_k - \overline{y}\|$ is monotonically decreasing. This is because that for any sequence $\{y_k\}$ that converges to $\overline{y}$, there exists a subsequence $\{y_k\}_{k\in\mathcal{K}}$ such that $\|y_k - \overline{y}\|$ is monotonically decreasing for $k \in \mathcal{K}$. Such an assumption is needed in order to have the constructed set sequence $\{C_k\}$ to be nested.

P. 65 The set $X \cap C_k$ is nonempty for all k. To see this, per definition of y_k, there exists some $x_k \in X \cap C$ such that $Ax_k = y_k$. In addition, $x_k \in N_k$. Therefore, $x_k \in X \cap C \cap N_k$, which amounts to that the set $X \cap C_k$ is nonempty.

P. 65 Judged solely by the proof here, it also suffices to have the condition $R_X \cap R_C \subset L_C$ hold, instead of the condition $R_X \cap R_C \cap N(A) \subset L_C$ in the proposition statement [COT Prop. 1.4.13, P. 64]. However, since $R_X \cap R_C \subset L_C$ always implies $R_X \cap R_C \cap N(A) \subset L_C$ while the vice versa does not hold, using $R_X \cap R_C \cap N(A) \subset L_C$ in the proposition covers broader situations, thus gives stronger results.

P. 66 This amounts to the case where the set sequence $\{C_k\}$ constructed in [COT Prop. 1.4.13 Proof, P. 65] are compact sets, in view of C_k being convex and [COT Prop. 1.4.2 (a), P. 45].

P. 68 Given that both $C_1, C_2 \subset \mathbb{R}^n$ are nonempty and $a \neq 0$, we have $\sup_{x\in C_1} a'x \in (-\infty, \infty]$, and $\inf_{x\in C_2} a'x \in [-\infty, \infty)$. If in addition, it holds that $\sup_{x\in C_1} a'x \le \inf_{x\in C_2} a'x$, then both $\sup_{x\in C_1} a'x$ and $\inf_{x\in C_2} a'x$ are real-valued.

P. 70

Lemma (1, P. 70)**.** *Let $S \subset \mathbb{R}^n$ be a subspace. Then S is closed.*

Proof. When $S = \mathbb{R}^n$, the result holds. Otherwise, denote as $b_1, \ldots, b_\ell$ a basis of S. For any convergent sequence $\{s^k\} \subset S$, we will show that its limit is in S. Since $s^k \in S$, then $s^k = \sum_{i=1}^{\ell} \alpha_i^k b_i$. Since $\{s^k\}$ is convergent, it is therefore bounded by some $M > 0$. Therefore, $\{\alpha_i^k\}_{k=0}^{\infty}$ is bounded by $M/\|b_i\|$ in view of $|\alpha_i^k| \cdot \|b_i\| \le \|s^k\|$. Therefore, the sequence $\{\alpha^k\} \subset \mathbb{R}^\ell$, where $\alpha^k = (\alpha_1^k, \ldots, \alpha_\ell^k)$, is bounded and has limit point $\bar{\alpha}$, which is the limit of the subsequence $\{\alpha^k\}_{k\in\mathcal{K}}$. Denote as $\bar{s} = \sum_{i=1}^{\ell} \bar{\alpha}_i b_i$. Then we have $\{s^k\}_{k\in\mathcal{K}}$ converges to $\bar{s}$ in view of $\|s^k - \bar{s}\| \le \sum_{i=1}^{\ell} |\alpha_i^k - \bar{\alpha}_i| \cdot \|b_i\|$. Since the limit is unique, then we see that S is closed. $\square$

Lemma (2, P. 70)**.** *Let $A \subset \mathbb{R}^n$ be an affine set. Then A is closed.*

Proof. The proof is essentially the same as [COT Note Lemma 1, P. 70]. □

Lemma (3, P. 70)**.** *Let $X \subset \mathbb{R}^n$ be some nonempty set. If* $\text{int}(X) \neq \emptyset$*, then* $\text{aff}(X) = \mathbb{R}^n$.

Proof. Since $\text{int}(X) \neq \emptyset$, then there exists some $\bar{x} \in X$ and $\varepsilon > 0$ such that $B_\varepsilon \subset X$, where $B_\varepsilon = \{x \mid \|x - \bar{x}\| < \varepsilon\}$. Denote as $e_1, \ldots, e_n$ a set of unit length basis of $\mathbb{R}^n$. Then it is clear that $\bar{x} + \nu e_i \in B_\varepsilon$ $i = 1, \ldots, n$ for some $\nu < \varepsilon$. Therefore, $\bar{x} + \text{span}(e_1, \ldots, e_n) \subset \text{aff}(X) \subset \mathbb{R}^n$. In view of the facts that $\text{span}(e_1, \ldots, e_n) = \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$, we conclude the proof. □

Lemma (4, P. 70)**.** *Let $C \subset \mathbb{R}^n$ be some nonempty convex set. Then* $\text{int}(C) = \emptyset$ *if and only if* $\text{aff}(C) \neq \mathbb{R}^n$

Proof. The if part is proven by [COT Note Lemma 3, P 70]. For the only if part, given $\text{int}(C) = \emptyset$, assume $\text{aff}(C) = \mathbb{R}^n$. Then by definition of relative interior, we see that $\text{int}(C) = \text{ri}(C)$. On the other hand, by [COT Prop. 1.3.2 (a), P. 24], we have $\text{ri}(C) \neq \emptyset$, which is a contradiction. Therefore, we have $\text{aff}(C) \neq \mathbb{R}^n$. □

Lemma (5, P. 70)**.** *Let $C \subset \mathbb{R}^n$ be a nonempty convex set. Then* $\text{int}(C) = \text{int}\big(\text{cl}(C)\big)$.

Proof. If $\text{int}(C) \neq \emptyset$, then by [COT Note Lemma 3, P. 70], we have $\text{aff}(C) = \mathbb{R}^n$ and $\text{ri}(C) = \text{int}(C)$ per definition. In addition, since $\text{aff}\big(\text{cl}(C)\big) = \text{aff}(C)$ [COAe1 Ex 1.18(a), P. 20], we also have $\text{ri}\big(\text{cl}(C)\big) = \text{int}\big(\text{cl}(C)\big)$. In view of [COT Prop. 1.3.5 (b), P. 28], we have $\text{int}(C) = \text{int}\big(\text{cl}(C)\big)$.

If $\text{int}(C) = \emptyset$, then by the only if part of [COT Note Lemma 4, P. 70], $\text{aff}(C) \neq \mathbb{R}^n$. Since $\text{aff}\big(\text{cl}(C)\big) = \text{aff}(C)$, we have $\text{aff}\big(\text{cl}(C)\big) \neq \mathbb{R}^n$. Then again by the if part of [COT Note Lemma 4, P. 70] and due to $\text{cl}(C)$ being convex, we have $\text{int}\big(\text{cl}(C)\big) = \emptyset$. □

Note that in [Lemmas 4, 5, P. 70], convexity is needed. For an example where X is nonempty, and nonconvex and has empty interior, while its closure has nonempty closure, consider $X \subset \mathbb{R}$ as the set of all rational numbers. Then its closure is $\mathbb{R}$, which has nonempty interior.

P. 73

Lemma (P. 73)**.** *Let X_1 and X_2 be two nonempty sets. Then X_1 and X_2 can be properly separated if and only if there exists some a such that*

$$\sup_{x \in X_1} a'x \leq \inf_{x \in X_2} a'x, \quad \inf_{x \in X_1} a'x < \sup_{x \in X_2} a'x. \tag{45}$$

Proof. If X_1 and X_2 can be properly separated, then there exists some nonzero a and scalar b that define a hyperplane H such that $\sup_{x\in X_1} a'x \le b \le \inf_{x\in X_2} a'x$. In addition, at least one of the two sets is not contained in H. Assume $X_1 \not\subset H$, then there exists $x \in X_1$ such that $a'x < b$, which has the strict inequality hold. The case where $X_2 \not\subset H$ can be similarly argued.

If Eq. (45) hold, we first note that a is nonzero. In addition, by the arguments given in [COT Note P. 68], we see that $\sup_{x\in X_1} a'x$, $\inf_{x\in X_2} a'x$ are both real numbers. Denote as b the value $\sup_{x\in X_1} a'x$. We will show that the hyperplane H defined by a and b properly separate X_1 and X_2. Per definition, we see that H separates X_1 and X_2. If H contains both X_1 and X_2, then the strict inequality in Eq. (45). Therefore, H properly separates X_1 and X_2. □

P. 73 Let C_1 and C_2 be two nonempty convex sets such that $C_1 \cap C_2 = \emptyset$ and $\mathrm{aff}(C_1 \cup C_2) = \mathbb{R}^n$. Let hyperplane H separate C_1 and C_2. If H does not properly separate C_1 and C_2, then $C_1 \cup C_2 \subset H$ where H is affine and is of $n-1$ dimension, which is a contradiction of $\mathrm{aff}(C_1 \cup C_2) = \mathbb{R}^n$. Therefore, the assumption is false and every hyperplane that separates C_1 and C_2 must properly separate them.

75

Lemma (1, P. 75)**.** *Let X_1, X_2 be two nonempty sets of $\mathbb{R}^n$. Then it holds that $\mathrm{aff}(X_1 + X_2) = \mathrm{aff}(X_1) + \mathrm{aff}(X_2)$.*

Proof. Denote the dimensions of $\mathrm{aff}(X_1)$, $\mathrm{aff}(X_2)$ as m_1 and m_2 respectively. Then by [COTe1 Ex 1.11 (b), P. 17], we have, for some $x_1^1, \ldots, x_{m_1}^1, \bar{x}^1 \in X_1$, and $x_1^2, \ldots, x_{m_2}^2, \bar{x}^2 \in X_2$,

$$\begin{aligned}
\mathrm{aff}(X_1) &= \Big\{ y \,\Big|\, y = \sum_{i=1}^{m_1} \alpha_i^1 (x_i^1 - \bar{x}^1) + \bar{x}^1 \Big\}, \\
\mathrm{aff}(X_2) &= \Big\{ y \,\Big|\, y = \sum_{i=1}^{m_2} \alpha_i^2 (x_i^2 - \bar{x}^2) + \bar{x}^2 \Big\}.
\end{aligned} \tag{46}$$

For any $y^1 + y^2 \in X_1 + X_2$, we have $y^1 + y^2 \in \mathrm{aff}(X_1) + \mathrm{aff}(X_2)$. In addition, $\mathrm{aff}(X_1) + \mathrm{aff}(X_2)$ is affine (one can verify this by using the definition of affine set). Therefore, $\mathrm{aff}(X_1 + X_2) \subset \mathrm{aff}(X_1) + \mathrm{aff}(X_2)$.

As for the reverse direction, for any $y^1 + y^2 \in \mathrm{aff}(X_1) + \mathrm{aff}(X_2)$, we have, by (46),

$$\begin{aligned}
y^1 &= \sum_{i=1}^{m_1} \beta_i (x_i^1 - \bar{x}^1) + \bar{x}^1, \\
y^2 &= \sum_{i=1}^{m_2} \gamma_i (x_i^2 - \bar{x}^2) + \bar{x}^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned} y^1 + y^2 &= \sum_{i=1}^{m_1} \beta_i\big((x_i^1 + \bar{x}^2) - (\bar{x}^1 + \bar{x}^2)\big) + \sum_{i=1}^{m_2} \gamma_i\big((\bar{x}^1 + x_i^2) - (\bar{x}^1 + \bar{x}^2)\big) + (\bar{x}^1 + \bar{x}^2) \\ &= \sum_{i=1}^{m_1} \beta_i(x_i^1 + \bar{x}^2) + \sum_{i=1}^{m_2} \gamma_i(\bar{x}^1 + x_i^2) + \big(1 - \sum_{i=1}^{m_1} \beta_i - \sum_{i=1}^{m_2} \gamma_i\big)(\bar{x}^1 + \bar{x}^2), \end{aligned}$$

which is affine combination of elements in $X_1 + X_2$. Therefore, the proof is complete. □

By above result, we have $\mathrm{aff}(\hat{C}) = \mathrm{aff}(C) + S^\perp = \mathbb{R}^n$. Since $\hat{C}$ is convex, then per definition we have $\mathrm{int}(\hat{C}) = \mathrm{ri}(\hat{C})$, which, due to [COT Prop. 1.3.2 (a), P. 24], is nonempty.

P. 75

Lemma (2, P. 75)**.** *Let $A_1, A_2 \subset \mathbb{R}$ be two sets of scalars, and $B = A_1 + A_2$. Then it holds that* $\sup A_1 + \sup A_2 = \sup B$.

Proof. Denote $\bar{a}_i = \sup A_i$, $i = 1, 2$, $\bar{a} = \bar{a}_1 + \bar{a}_2$, and $\bar{b} = \sup B$. Then we have

$$\forall b\big(b \in B \iff \exists a_1 \in A_1 \exists a_2 \in A_2(b = a_1 + a_2) \implies b \le \bar{a}\big).$$

Therefore, we have $\bar{b} \le \bar{a}$.

Next, we will show that $\bar{b} < \bar{a}$ cannot hold. For the case $\bar{a} = \infty$, we can see that B is also unbounded above and therefore $\bar{b} = \infty$. Otherwise, $\bar{a} < \infty$. If $\bar{b} < \bar{a}$, then there exists some $\varepsilon > 0$ such that $\bar{a} - 2\varepsilon > \bar{b}$. Then there exists $a_i \in A_i$ such that $a_i > \bar{a}_i - \varepsilon$ for $i = 1, 2$. This implies $a_1 + a_2 \in B$ and $a_1 + a_2 > \bar{b}$, which is a contradiction. Therefore, we have $\bar{b} = \bar{a}$. □

Note that for $A_1, A_2 \subset \mathbb{R}$ being two sets of scalars, and $B = A_1 + A_2$, we also have $\inf A_1 + \inf A_2 = \inf B$.

Here, by setting $B = \{a'x \,|\, x \in \hat{C}\}$, $A_1 = \{a'x \,|\, x \in C\}$, $A_2 = \{a'z \,|\, z \in S^\perp\}$, we get the desired result.

P. 76

Lemma (P. 76)**.** *Let A_1, A_2 be two nonempty sets. Then $0 \in A_1 - A_2$ if and only if $A_1 \cap A_2 \neq \emptyset$.*

Proof. For the only if part, given $0 \in A_1 - A_2$, then $\exists a_1 \in A_1 \exists a_2 \in A_2(a_1 = a_2)$, which implies $\exists a_1 \in A_1 \cap A_2$, which is equivalent to $A_1 \cap A_2 \neq \emptyset$.

For the if part, given $A_1 \cap A_2 \neq \emptyset$, then $\exists a_1 \in A_1 \cap A_2$. Since $\forall a \exists b(a = b)$, then $\exists a_1 \in A_1 \exists a_2 \in A_2(a_1 - a_2 = 0)$, which per definition, means $0 \in A_1 - A_2$. □

P. 76 Apart from [COT Note Lemma P. 76], it also applies [COT Note Lemmas P. 73, Lemma 2, P. 75], and the fact that for a set $A \subset \mathbb{R}$, $\inf A = \sup(-A)$.

P. 78 To see this, note that

$$\overline{x} \in P \cap \mathrm{ri}(\overline{C}) \implies \overline{x} \in P \cap \mathrm{ri}(\overline{C}) \cap \overline{C} \implies \overline{x} \in P \cap \mathrm{ri}(\overline{C}) \cap \mathrm{aff}(C).$$

P. 78

Lemma (1, P. 78)**.** *Let $P = \{x \,|\, a_j' x \leq b_j,\ j = 1, \ldots, m\} \subset \mathbb{R}^n$ be a polyhedron, and $\bar{x} \in \mathbb{R}^n$. Then $P = \bar{x} + \bar{P}$, where $\bar{P} = \{x \,|\, a_j' x \leq b_j - a_j' \bar{x},\ j = 1, \ldots, m\}$.*

Proof. For $x \in P$, we have $a_j' x \leq b_j$ for all j, which implies $a_j'(x - \bar{x}) \leq b_j - a_j' \bar{x}$, namely $x - \bar{x} \in \bar{P}$. Therefore, we have $x \in \bar{x} + \bar{P}$.

Conversely, $x \in \bar{x} + \bar{P}$ implies $x - \bar{x} \in \bar{P}$, which indicates that $a_j' x \leq b_j$ for all j, namely $x \in P$. □

P. 78

Lemma (2, P. 78)**.** *$P = \{x \,|\, a_j' x \leq b_j,\ j = 1, \ldots, m\} \subset \mathbb{R}^n$ be a polyhedron. If $a_j' \bar{x} < b_j$ for all $j = 1, \ldots, m$, then $\bar{x} \in \mathrm{int}(P)$.*

Proof. Denote as F_j the closed half space $\{x \,|\, a_j' x \leq b_j\}$. Then it is clear that $P = \cap_{j=1}^m F_j$. We will show that for every F_j, there exists an $\varepsilon_j > 0$ such that the closed ball with center $\bar{x}$ and radius ε_j is fully contained in F_j, then the ball with center $\bar{x}$ and radius $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_m\}$ is then fully contained in P, which shows $\bar{x}$ being an interior point.

We take F_1 as an example. Since $a_1' \bar{x} < b_1$, then the distance from $\bar{x}$ to the hyperplane $H_1 = \{x \,|\, a_1' x = b_1\}$ is $d_1 = |a_1' \bar{x} - b_1| / \|a_1\|$. Then for all v with $\|v\| \leq d_1$, we have $\bar{x} + v \in F_1$. To see this, note that

$$a_1'(\bar{x} + v) \leq a_1' \bar{x} + |a_1' v| \leq \|a_1\| \cdot \|v\| + a_1' \bar{x} \leq |b_1 - a_1' \bar{x}| + a_1' \bar{x} = b_1,$$

where the second inequality is by Cauchy–Schwarz inequality, the third inequality is due to $\|v\| \leq d_1$, and the equality is due to $\bar{x} \in F_1$. Therefore, by setting $\varepsilon_1 = d_1$, we have the desired result. □

We fill in some details for the followup arguments. If 0 is an interior point of P, then by [COT Note Lemma 3, P. 70], we have $\mathrm{aff}(P) = \mathbb{R}^n$ and $\mathrm{ri}(P) = \mathrm{int}(P)$. Then since $0 \in \mathrm{ri}(P) \cap \overline{C} \subset \mathrm{ri}(P) \cap \mathrm{aff}(C)$, while $\mathrm{ri}\big(\mathrm{aff}(C)\big) = \mathrm{aff}(C)$, we have $\mathrm{ri}(P) \cap \mathrm{ri}\big(\mathrm{aff}(C)\big) \neq \emptyset$. Therefore, by [COT Note Lemma 3, P. 32], we have $\mathrm{aff}(D) = \mathrm{aff}\big(P \cap \mathrm{aff}(C)\big) = \mathrm{aff}(P) \cap \mathrm{aff}\big(\mathrm{aff}(C)\big) = \mathrm{aff}(C)$. In addition, by [COT Prop. 1.3.8, P. 32], we have $0 \in \mathrm{ri}(P) \cap \mathrm{ri}\big(\mathrm{aff}(C)\big) = \mathrm{ri}\big(P \cap \mathrm{aff}(C)\big) = \mathrm{ri}(D)$. Then for any $\overline{x} \in \mathrm{ri}(\overline{C}) \subset \mathrm{aff}(C)$, the line defined by 0 and $\overline{x}$ belongs to $\mathrm{aff}(C) = \mathrm{aff}(D)$. Due to $0 \in \mathrm{ri}(D)$, then there exists some $\varepsilon > 0$, such that

$\varepsilon\overline{x} \in D$, then by Line Segment Principle, every point between 0 and $\varepsilon\overline{x}$ belongs to $\mathrm{ri}(D)$.

P. 79

Lemma (1, P. 79)**.** *Let $P = K \cap Q$ where $K = \{x \,|\, a_j' x \le 0,\ j = 1, \ldots, m\} \subset \mathbb{R}^n$ and $Q = \{x \,|\, a_j' x \le b_j,\ b_j > 0,\ j = m+1, \ldots, \bar{m}\} \subset \mathbb{R}^n$. Then $K = \mathrm{cone}(P)$.*

Proof. Let $x \in \mathrm{cone}(P)$, then $x = \sum_{i=1}^{\ell} \alpha^i x^i$ with $x^i \in P$ and $\alpha \ge 0$. Then for $j = 1, \ldots, m$, $a_j' x = a_j' \sum_{i=1}^{\ell} \alpha^i x^i \le 0$, indicating that $x \in K$.

Conversely, if $x \in K$ and x is nonzero, then $a_j' x \le 0$ for $j = 1, \ldots, m$. In addition, due to [COT Note Lemma 2, P. 78], we have $0 \in \mathrm{int}(Q)$. Then $\exists \varepsilon > 0$ such that $\|v\| \le \varepsilon$ implies $v \in Q$. Therefore, we have $(\varepsilon/\|x\|)x \in Q \cap K = P$. Then we have $x = \alpha(\varepsilon/\|x\|)x$ where $\alpha = \|x\|/\varepsilon > 0$, which implies $x \in \mathrm{cone}(P)$. ☐

P. 79

Lemma (2, P. 79)**.** *Let $A \subset \mathbb{R}^{m+n+\ell}$ be a nonempty set with elements of the form (x, y, z), $P_{XY} = \{(x, y) \,|\, \exists z \in \mathbb{R}^{\ell} \text{ such that } (x, y, z) \in A\}$, $P_X = \{x \,|\, \exists y \in \mathbb{R}^n \exists z \in \mathbb{R}^{\ell} \text{ such that } (x, y, z) \in A\}$, and $\bar{P}_X = \{x \,|\, \exists y \in \mathbb{R}^n \text{ such that } (x, y) \in P_{XY}\}$. Then $P_X = \bar{P}_X$.*

Proof.

$$
\begin{aligned}
x \in P_X &\iff \exists y \in \mathbb{R}^n \exists z \in \mathbb{R}^{\ell} \text{ such that } (x, y, z) \in A \\
&\iff \exists y \in \mathbb{R}^n \text{ such that } (x, y) \in P_{XY} \iff x \in \bar{P}_X.
\end{aligned}
$$

☐

Above result shows that projection step by step is equivalent to projection all together.

Lemma (3, P. 79)**.** *Let $P \subset \mathbb{R}^{n+m}$ be the polyhedron set $\{(x, y) \,|\, a_j' x + c_j' y \le b_j,\ j = 1, \ldots, \ell\}$. Then $P_X = \{x \,|\, \exists y \in \mathbb{R}^m \text{ such that } (x, y) \in P\}$ is a polyhedron.*

Proof. We consider $m = 1$ and for $m > 1$, the proof can be done by induction in view of [COT Note Lemma 2, P. 79].

Denote $J = \{1, 2, \ldots, \ell\}$, $J_0 = \{j \,|\, j \in J,\ c_j = 0\}$, $J_+ = \{j \,|\, j \in J,\ c_j > 0\}$, and $J_- = \{j \,|\, j \in J,\ c_j < 0\}$. In what follows, we apply the Fourier-Motzkin elimination to construct a projection set. For $(x, y) \in P$, we have

$$
\begin{aligned}
&a_k' x \le b_k, && k \in J_0, \\
&(a_j' x)/c_j \le b_j/c_j - y \iff -(a_j' x)/c_j + b_j/c_j \ge y, && j \in J_+, \\
&(a_i' x)/c_i \ge b_i/c_i - y \iff -(a_i' x)/c_i + b_i/c_i \le y, && i \in J_-.
\end{aligned}
$$

Denote as $\bar{a}_j = -a_j/c_j$, and $\bar{b}_j = b_j/c_j$ for $j \in J_+ \cup J_-$, and define the set $\bar{P}_X$ given as

$$\bar{P}_X = \{x \mid \bar{a}_j'x + \bar{b}_j \geq \bar{a}_i'x + \bar{b}_i,\, j \in J_+,\, i \in J_-,\, a_k'x \leq b_k,\, k \in J_0\}.$$

We claim $P_X = \bar{P}_X$. To see this, if $x \in P_X$, then by above computation and definition of $\bar{P}_X$, we have $x \in \bar{P}_X$. Conversely, if $x \in \bar{P}_X$, then $\min_{j\in J_+} \bar{a}_j'x + \bar{b}_j \geq \max_{i\in J_-} \bar{a}_i'x + \bar{b}_i$. Define $y = \min_{j\in J_+} \bar{a}_j'x + \bar{b}_j$, and we have $(x, y) \in P$, implying that $x \in P_X$. □

Lemma (4, P. 79)**.** *Let P_1 and P_2 be two polyhedral sets of $\mathbb{R}^n$. Then $P = P_1 + P_2$ is a polyhedron.*

Proof. We define the set $\bar{P} = \{(x, y, z) \mid x \in P_1,\, y \in P_2,\, z = x + y\}$. It is clear that $\bar{P}$ is a polyhedron of $\mathbb{R}^{3n}$. In addition, define $\bar{P}_Z = \{z \mid \exists x \in \mathbb{R}^n \exists y \in \mathbb{R}^n$ such that $(x, y, z) \in \bar{P}\}$. By [COT Note Lemma 3, P. 79], we know $\bar{P}_Z$ is a polyhedron. In addition, per definition, we see that $\bar{P}_Z = P$. Therefore, P is a polyhedron. □

P. 79 Note that the set K is polyhydron and also a cone. Then by [COT Note Lemma, P. 4], we see that the halfspaces used to define K pass through 0.

P. 79

Lemma (5, P. 79)**.** *Let $H = \{x \mid a'x = 0\}$ be a hyperplane in $\mathbb{R}^n$ that defines a closed half space $F = \{x \mid a'x \leq 0\}$. Let S be a subspace such that $S \not\subset H$ and $S \cap H \neq \emptyset$. Denote as A the set $S \cap F$ and let $\bar{F}$ be a closed half space $\{x \mid \bar{a}'x \leq 0\}$ such that $S \cap H \subset \bar{F}$. Then if $\mathrm{ri}(A) \cap \bar{F} \neq \emptyset$, it holds that $A \subset \bar{F}$.*

Proof. First, in view of [COT Note Lemma 3, P. 70, Lemma 2, P. 78], we have $\mathrm{ri}(F) = \mathrm{int}(F) = \{x \mid a'x < 0\}$. Besides, per definition, $\mathrm{ri}(S) = S$. Since $S \not\subset H$, then there exists $s \in S$ such that $a's < 0$, namely $s \in \mathrm{ri}(F)$. Therefore, $\mathrm{ri}(F) \cap \mathrm{ri}(S) \neq \emptyset$. Then, in view of [COT Prop. 1.3.8, P. 32], we have $\mathrm{ri}(A) = \mathrm{ri}(F) \cap \mathrm{ri}(S) = \{x \mid a'x < 0,\, x \in S\}$.

Denote as ℓ the dimension of $H \cap S$, which is a subspace (or origin) in view that H and S are both subspaces. Denote as $V = \{v_1, \ldots, v_\ell\}$ a set of orthogonal basis of $H \cap S$. Then $\mathrm{span}(v_1, \ldots, v_\ell) \subset H$ and thus $a'v_j = 0$ for $j = 1, \ldots, \ell$. (ℓ could be 0, in which case V is an empty set.) Denote as $V \cup U$ an orthogonal basis of H, where $U = \{u_1, \ldots, u_{n-\ell-1}\}$ and thus $V \cup U \cup \{a\}$ forms an orthogonal basis of $\mathbb{R}^n$. In addition, let S have dimension m, and denote as $V \cup W$ an orthogonal basis of S where $W = \{w_1, \ldots, w_{m-\ell}\}$. We will show that the set W is singleton, and $\ell = m - 1$. First, we note that W is not empty since $S \not\subset H$. In addition, $a'w \neq 0$ for $w \in W$, as otherwise, we have $w \in S \cap H$, contradicting that $\mathrm{span}(v_1, \ldots, v_\ell) = S \cap H$. Second, if W has more than one element, say

w_1 and w_2, then

$$w_1 = \sum_{i=1}^{n-\ell-1} \alpha_i u_i + \beta_1 a, \ w_2 = \sum_{i=1}^{n-\ell-1} \gamma_i u_i + \beta_2 a,$$

in view that w_1, w_2 are orthogonal to V, and β_1, β_2 are nonzero since $a'w \neq 0$ for $w \in W$ and $a'u = 0$ for $u \in U$. On the other hand, we have $\beta_2 w_1 - \beta_1 w_2 \in S$ since S is a subspace, $\beta_2 w_1 - \beta_1 w_2$ is nonzero since w_1 and w_2 are linearly independent, and $\beta_2 w_1 - \beta_1 w_2 \in H$ since it is linear combination of U. Thus $\beta_2 w_1 - \beta_1 w_2 \in S \cap H$. However, this is a contradiction to $\text{span}(v_1, \ldots, v_\ell) = S \cap H$ since $\beta_2 w_1 - \beta_1 w_2$ is nonzero and also orthogonal to V thus linearly independent. Therefore, the assumption is false and W is singleton. This means $S = \text{span}(v_1, \ldots, v_\ell, w_1)$ and $\ell = m - 1$.

Now we are ready to prove the main result. Since $S \cap H \subset \bar{F}$, then $S \cap H \subset L_{\bar{F}}$, namely $\bar{a}'v = 0$ for $v \in V$. For every $\bar{z} \in \text{ri}(A)$, it holds that

$$\bar{z} = \sum_{i=1}^{\ell} \bar{\kappa}_i v_i + \bar{\eta} w_1,$$

and we have $\bar{\eta} a' w_1 < 0$ with $\bar{\eta} \neq 0$ since $\bar{z} \in \text{ri}(A)$. If there exists a $\bar{z} \in \text{ri}(A)$ that is also in $\bar{F}$, then we have $\bar{\eta}\bar{a}'w_1 \leq 0$. Then for every $z \in \text{ri}(A)$ where

$$z = \sum_{i=1}^{\ell} \kappa_i v_i + \eta w_1,$$

it holds that $\eta a' w_1 < 0$, $\eta/\bar{\eta} > 0$. Thus, we have $\bar{a}'z = \eta \bar{a}' w_1 = (\eta/\bar{\eta})\bar{\eta}\bar{a}'w_1 \leq 0$ as $\eta/\bar{\eta} > 0$, $\bar{\eta}\bar{a}'w_1 \leq 0$. This means $z \in \bar{F}$. □

P. 79

Lemma (6, P. 79)**.** *Let C be a nonempty convex set that is contained in a closed half space F whose corresponding hyperplane is denoted as H. In addition, C is not contained in H. Denote as $\bar{C}$ the set* $\text{aff}(C) \cap F$*. Then* $\text{ri}(C) \subset \text{ri}(\bar{C})$.

Proof. First, we note that $\text{ri}(F) = F \setminus H$. Since $C \subset F$ and $C \not\subset H$, then there exists $x \in C$ such that $x \in \text{ri}(F)$. Therefore, $\text{ri}(F) \cap \text{ri}\big(\text{aff}(C)\big) = \text{ri}(F) \cap \text{aff}(C) \neq \emptyset$. Thus, we have $\text{ri}(\bar{C}) = \text{ri}(F) \cap \text{aff}(C)$, per [COT Prop. 1.3.8, P. 32]. On the other hand, since $C \not\subset H$, $C \subset F$, and $\text{ri}(F) = F \setminus H$, by [COT Eq. (1.33), P. 77], we have $\text{ri}(C) \subset \text{ri}(F)$. On the other hand, $\text{ri}(C) \subset \text{aff}(C)$. Therefore, $\text{ri}(C) \subset \text{ri}(F) \cap \text{aff}(C)$. □

P. 79

Lemma (7, P. 79)**.** *Let A, B be two nonempty subsets of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then*

$$(A + x) \cap (B + x) = (A \cap B) + x. \tag{47}$$

Proof. If $A \cap B = \emptyset$, one can verify that both sides of Eq. (47) are emptysets. Otherwise, assume $A \cap B \neq \emptyset$. Then we have

$$\begin{aligned} y \in (A+x) \cap (B+x) &\implies (y \in A+x) \wedge (y \in B+x) \\ &\implies (y-x \in A) \wedge (y-x \in B) \\ &\implies y-x \in (A \cap B) \\ &\implies y \in (A \cap B)+x. \end{aligned}$$

Conversely, we have

$$\begin{aligned} y \in (A \cap B)+x &\implies y-x \in (A \cap B) \\ &\implies (y-x \in A) \wedge (y-x \in B) \\ &\implies (y \in A+x) \wedge (y \in B+x) \\ &\implies y \in (A+x) \cap (B+x). \end{aligned}$$

□

We rewrite the critical steps of the proof arguments replacing the part starting from 'We thus assume that $P \cap \overline{C} \neq \emptyset$, and by using ...'.

Assume $P \cap \overline{C} \neq \emptyset$ and $y \in P \cap \overline{C}$. Then by using similar arguments as given in the proof, we can see that $P = y + \bar{P}$, where

$$\bar{P} = \{x \,|\, a_j' x \leq 0,\, j = 1, \ldots, m,\, a_i' x \leq b_i,\, b_i > 0,\, i = m+1, \ldots, \bar{m}\},$$

so that y is not an interior point of P.

Then denote $M = H \cap \text{aff}(C)$, and we have $y \in M$ since $y \in P \cap \overline{C} \subset \text{aff}(C)$ and also $y \in H$. Define $K = \text{cone}(\bar{P}) + M$. We claim $K \cap \text{ri}(\overline{C}) = \emptyset$ and proof is done by contradiction. Assume $\bar{x} \in K \cap \text{ri}(\overline{C})$. Then $\bar{x} = u + v$ with $u \in \text{cone}(\bar{P})$ and $v \in M$. u must be nonzero since otherwise we have $\bar{x} = v \in \text{ri}(\overline{C})$ while $v \in M \subset H$, which means $v \in \text{ri}(\overline{C}) \cap H$, contradicting with $\text{ri}(\overline{C}) \cap H = \emptyset$. In view of the proof of [COT Note Lemma 1, P. 79], we have $u = \alpha w$ with $\alpha > 0$ and $w \in \bar{P}$. In fact, α can be arbitrarily large, as indicated in the proof of [COT Note Lemma 1, P. 79]. It is clear that $w + y \in P$ since $w \in \bar{P}$. In what follows, we will show that $w + y \in \text{ri}(\overline{C})$. First, we note that

$$w + y = \bar{x}/\alpha - v/\alpha + y = (\bar{x} - y)/\alpha + y - (v - y)/\alpha.$$

Then since $(\bar{x} - y)/\alpha + y = \bar{x}/\alpha + (\alpha - 1)/\alpha y$, $\alpha > 0$ can be chosen to be arbitrarily large such that $1/\alpha$, $(\alpha - 1)/\alpha \in (0, 1)$, $\bar{x} \in \text{ri}(\overline{C})$, $y \in \overline{C}$, then by Line Segment Principle, we see that $(\bar{x} - y)/\alpha + y \in \text{ri}(\overline{C})$. Since M is affine and $y \in M$, then $S = M - y$ is a subspace and $L_M = S$. Since $M \subset \overline{C}$ and $M \neq \emptyset$, then $S = L_M \subset L_{\overline{C}} = L_{\text{ri}(\overline{C})}$, by [COT Prop. 1.4.3 (b) (c), P. 47]. Then $v \in M$ yields $v - y \in S \in L_{\text{ri}(\overline{C})}$. Therefore, $w + y = (\bar{x} - y)/\alpha + y - (v - y)/\alpha \in \text{ri}(\overline{C})$. Thus we have shown that $w + y \in P \cap \text{ri}(\overline{C})$, contradicting $P \cap \text{ri}(\overline{C}) = \emptyset$. Therefore, $K \cap \text{ri}(\overline{C}) = \emptyset$.

Due to [COT Note Lemma 4, P. 79], $K = \text{cone}(\bar{P}) + S + y$ is a polyhedral set. In addition, denote as $\bar{K}$ the set $\text{cone}(\bar{P}) + S$, which is also polyhedral. Also, $\bar{K}$ is a cone per definition. Therefore, by [COT Note Lemma, P. 4], $\bar{K}$ can be defined by a set of half spaces $\bar{F}_1, \ldots, \bar{F}_r$ that passes through 0. Similarly, denote as $F_i = \bar{F}_i + y$, $i = 1, \ldots, r$. Clearly, we have $S \subset \bar{F}_i$, $M \subset F_i$. Then we apply [COT Note Lemma 5, P. 79], S in the lemma being $\text{aff}(C) - y$, A in the lemma being the set $\overline{C} - y$, the hyperplane H in the lemma being $H - y$ here, the set $\bar{F}$ in the lemma being $\bar{F}_i$ for any $i = 1, \ldots, r$ here. Then by [COT Note Lemma 7. P. 79], we have $S = M - y = H \cap \text{aff}(C) - y = (H - y) \cap \big(\text{aff}(C) - y\big)$. Then there exists some $\bar{F}_i$, say $\bar{F}_1$, that does not contain any relative interior points of $\overline{C} - y$. Then by [COT Note Lemma 3, P. 26], $F_1 \cap \text{ri}(\overline{C}) = \emptyset$. Therefore, the hyperplane defining F_1 separates K and $\overline{C}$ while does not contain $\overline{C}$. Since $P \subset K \subset F_1$, and $C \subset \overline{C}$, the proof is complete.

P. 79 Note that when $P \cap \overline{C} \neq \emptyset$, we have $P \cap \overline{C} \subset M = H \cap \text{aff}(C)$. To see this, for $x \in P \cap \overline{C}$, since $\overline{C} \subset \text{aff}(C)$, we have $x \in P \cap \overline{C} \cap \text{aff}(C)$. Thus, $x \in D \cap \overline{C}$. Since H properly seperate $\overline{C}$ and D, then $x \in H$. Thus, $x \in H \cap \text{aff}(C) = M$.

P. 80 First, we note that there is some hyperplane H that contains C in one of its closed half spaces. Since C does not contain any vertical line, then there exists $(u, w) \notin C$. Then by [COT Prop. 1.5.1, P. 69], there exists some H that contains C in one of its closed half space.

P. 81 Alternatively, we can directly argue about set C. Since $C \subset \text{cl}(C)$, it holds that $\overline{\mu}'u > \gamma > \overline{\mu}'\overline{u}$, $\forall (u, w) \in C$. Then by part (a), there exists some (μ, β) and γ with $\beta \neq 0$ such that $\mu'u + \beta w > \gamma$, $\forall (u, w) \in C$. Then similar to the proof, we can find some $\varepsilon > 0$ such that a hyperplane with normal $(\overline{\mu} + \varepsilon\mu, \varepsilon\beta)$ strictly separates C and $(\overline{u}, \overline{w})$.

P. 81 Alternative arguments for this part can be given as follows: By part (a), there exists some (μ, β) with $\beta \neq 0$ such that the closed half space F defined by (μ, β) contains C. Then by the definition of closure, we have $\text{cl}(C) \subset F$.

P. 82 To have the inequality hold, what we require is to have $\varepsilon > 0$ and

$$\overline{\gamma} + \varepsilon\gamma > \overline{\mu}'\overline{u} + \varepsilon(\mu'\overline{u} + \beta\overline{w}) \iff \overline{\gamma} - \overline{\mu}'\overline{u} > \varepsilon(\mu'\overline{u} + \beta\overline{w} - \gamma).$$

Divide ε on both sides and note that $\overline{\gamma} - \overline{\mu}'\overline{u} > 0$, then $(\overline{\gamma} - \overline{\mu}'\overline{u})/\varepsilon \uparrow \infty$ as $\varepsilon \downarrow 0$.

P. 83 Let $X \subset [-\infty, \infty]$. Then $\sup X = -\inf(-X)$. This is obvious when $X \subset [-\infty, \infty)$. When $\infty \in X$, the relation also holds since $-\infty \in (-X)$ and $-\inf(-X) = -(-\infty) = \infty = \sup X$.

P. 83 When $f(x) = \infty$ for all x, we have $f^\star(y) = \sup_{x \in \mathbb{R}^n}\{x'y - f(x)\} = \sup\{-\infty\} = -\infty$ for all y. If there exists some $\bar{x}$ such that $f(\bar{x}) = -\infty$, then $\bar{x}'y - f(\bar{x}) = \infty$. Then we have $f^\star(y) = \infty$ for all y. Therefore, when f is improper, $f^\star$ is constant $-\infty$, or ∞, both being closed convex. When f is proper, we have $f(x) > -\infty$ for all x and $\mathrm{dom}(f) \neq \emptyset$. Since for all y, we have $x'y - f(x) = -\infty$ for all $x \in \mathbb{R}^n \setminus \mathrm{dom}(f)$, then we have

$$\begin{aligned}\sup\{x'y - f(x) \mid x \in \mathbb{R}^n\} &= \sup\big(\{x'y - f(x) \mid x \in \mathrm{dom}(f)\} \cup \{-\infty\}\big) \\ &= \sup\{x'y - f(x) \mid x \in \mathrm{dom}(f)\}.\end{aligned}$$

Then by [COT Prop. 1.1.6, P. 13], we see that $f^\star$ is closed convex.

In addition, if there exists some $\bar{y}$ such that $f^\star(\bar{y}) = -\infty$, then $f^\star(y) = -\infty$ for all y. To see this, note that $f^\star(\bar{y}) = \sup_{x\in\mathbb{R}^n}\{x'\bar{y} - f(x)\} = -\infty$. Since $x'\bar{y} - f(x) \geq -\infty$ for all x, then $x'\bar{y} - f(x) = -\infty$, $\forall x$. Therefore, we have $f(x) = \infty + x'\bar{y} = \infty$ for all x due to $x'\bar{y} \in \mathbb{R}$. As a result of above discussion, we see $f^\star(y) = -\infty$ for all y.

P. 86 Note that f being proper does not imply that $\check{\mathrm{cl}}\, f$ is proper. To see this, we consider an example $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} -\frac{1}{|x|}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then we can see that the vertical line through origin belongs to $\mathrm{conv}\big(\mathrm{epi}(f)\big)$. In fact, $\mathrm{conv}\big(\mathrm{epi}(f)\big) = \mathbb{R}^2$. Since $\mathrm{epi}(\check{\mathrm{cl}}\, f) = \mathrm{cl}\big(\mathrm{conv}\big(\mathrm{epi}(f)\big)\big)$, then we see that $\check{\mathrm{cl}}\, f$ is improper.

P. 86 Since $\mathrm{epi}(f)$ does not contain vertical lines, then by Nonvertical Hyperplane Theorem, there exists some μ, $\beta \neq 0$ and α, such that $\mu'x + \beta w \geq \alpha$ for all $(x, w) \in \mathrm{epi}(f)$. Since w can be arbitrarily large while the inequality still holds, we have $\beta > 0$. Thus by setting $y = \mu/\beta$, and $c = \alpha/\beta$, we get the asserted relation.

P. 87 Since f is proper, then $\mathrm{dom}(f) \neq \emptyset$ and $y'z - f(z) \in \mathbb{R}$ for all $z \in \mathrm{dom}(f)$. In addition, $y'z - f(z) = -\infty$ for all $z \in \mathbb{R}^n \setminus \mathrm{dom}(f)$. Thus, $\sup_{z\in\mathrm{dom}(f)}\{y'z - f(z)\} = \sup_{z\in\mathbb{R}^n}\{y'z - f(z)\}$.

P. 87 In fact, what is claimed here requires the following holds:

$$\inf_{x\in\mathbb{R}^n}\{g(x) - x'y\} = \sup\{c \in \mathbb{R} \mid w - x'y \geq c,\ \forall (x, w) \in \mathrm{epi}(g)\}. \tag{48}$$

When $g(x) = \infty$ for all $x \in \mathbb{R}^n$, we have $\mathrm{epi}(g) = \emptyset$. Thus, $\inf_{x\in\mathbb{R}^n}\{g(x) - x'y\} = \inf\{\infty\}$, and $\sup\{c \in \mathbb{R} \mid w - x'y \geq c,\ \forall (x, w) \in \mathrm{epi}(g)\} = \sup \mathbb{R} = \infty$. Namely, Eq. (48) hold.

When $g(\bar{x}) = -\infty$ for some $\bar{x}$, we have $\inf_{x\in\mathbb{R}^n}\{g(x) - x'y\} = -\infty$. On the other hand, $\{c \in \mathbb{R} \mid w - x'y \geq c,\ \forall (x, w) \in \mathrm{epi}(g)\} = \emptyset$ since $w - \bar{x}'y$ can be made arbitrarily small and $\sup \emptyset = -\infty$. Thus, what is left to show is when g is proper.

Indeed, given that g is proper, we have $\forall c \in \mathbb{R}$,

$$\begin{aligned} w - x'y \geq c,\ \forall (x, w) \in \mathrm{epi}(g) &\iff g(x) - x'y \geq c,\ \forall x \in \mathbb{R}^n \\ &\iff c \leq \inf_{x\in\mathbb{R}^n}\{g(x) - x'y\}. \end{aligned}$$

Thus, we have

$$\{c \in \mathbb{R} \mid w - x'y \geq c,\ \forall (x,w) \in \operatorname{epi}(g)\} = \Big\{c \in \mathbb{R} \mid c \leq \inf_{x \in \mathbb{R}^n} \{g(x) - x'y\}\Big\}.$$

When $\inf_{x\in\mathbb{R}^n}\{g(x) - x'y\} = -\infty$, we see the set above is empty and thus Eq. (48) holds. Otherwise, $\inf_{x\in\mathbb{R}^n}\{g(x) - x'y\} < \infty$ since g is proper. Then we have

$$\sup \Big\{c \in \mathbb{R} \mid c \leq \inf_{x \in \mathbb{R}^n} \{g(x) - x'y\}\Big\} = \inf_{x \in \mathbb{R}^n} \{g(x) - x'y\}.$$

What is argued in the proof is that

$$\{c \in \mathbb{R} \mid w - x'y \geq c,\ \forall (x,w) \in \operatorname{epi}(f)\} = \{c \in \mathbb{R} \mid w - x'y \geq c,\ \forall (x,w) \in \operatorname{epi}(\check{\operatorname{cl}}\, f)\}.$$

Then in view of Eq. (48), we have $f^\star(y) = \check{f}^\star(y)$.

P. 88 Let f be a closed proper convex function. Let H_N denote the intersection of all closed halfspaces that contain $\operatorname{epi}(f)$ and have nonvertical corresponding hyperplanes. We will use the arguments given in [COT Prop. 1.5.4, Proof, P. 73] and apply [COT Prop. 1.5.8 (b), P. 80] to show the assertion. First, by [COT Prop. 1.5.8 (a), P. 80], we know the closed halfspaces that contain $\operatorname{epi}(f)$ and have nonvertical corresponding hyperplanes do exist. Then by intersection, we have $\operatorname{epi}(f) \subset H_N$. Conversely, if $(x,w) \notin \operatorname{epi}(f)$, since f is closed, then by [COT Prop. 1.5.8 (b), P. 80], there exists a nonvertical hyperplane that strictly separates (x,w) and $\operatorname{epi}(f)$. Thus, $(x,w) \notin H_N$. Therefore, we see that $H_N = \operatorname{epi}(f)$.

P. 89 We can see that $\operatorname{cl} \delta_C$ is proper due to δ_C being proper and [COT Prop. 1.3.15 (a), P. 40]. In particular, we have the following lemmas hold.

Lemma (1, P. 89)**.** *Let $C \subset \mathbb{R}^n$ be a nonempty convex set and δ_C be its indicator function. Then the closure of δ_C, viz., $\operatorname{cl}\delta_C$, is equal to $\delta_{\operatorname{cl}(C)}$.*

Proof. We first show that $\operatorname{dom}(\operatorname{cl}\delta_C) = \operatorname{cl}(C)$. Due to [COT Prop. 1.3.15 (a), P. 40], we have $\operatorname{cl}\big(\operatorname{dom}(\operatorname{cl}\delta_C)\big) = \operatorname{cl}\big(\operatorname{dom}(\delta_C)\big) = \operatorname{cl}(C)$. Thus, $\operatorname{dom}(\operatorname{cl}\delta_C) \subset \operatorname{cl}(C)$. Also by the same proposition, we have $\operatorname{cl}\delta_C(x) = \delta_C(x) = 0\ \forall x \in \operatorname{ri}(C)$. Thus, we have $\operatorname{ri}(C) \subset V_0$, where V_0 is the 0-level set of $\operatorname{cl}\delta_C$. Per definition, we have $V_0 \subset \operatorname{dom}(\operatorname{cl}\delta_C)$. Since $\operatorname{cl}\delta_C$ is closed, by [COT Prop. 1.1.2, P. 10], V_0 is closed, and as a result, we have $\operatorname{cl}\big(\operatorname{ri}(C)\big) \subset V_0$. Put the above relations together and we have

$$\operatorname{cl}\big(\operatorname{ri}(C)\big) \subset V_0 \subset \operatorname{dom}(\operatorname{cl}\delta_C) \subset \operatorname{cl}(C).$$

By [COT Prop. 1.3.5 (a), P. 28], we also have $\operatorname{cl}\big(\operatorname{ri}(C)\big) = \operatorname{cl}(C)$. Thus, $V_0 = \operatorname{dom}(\operatorname{cl}\delta_C) = \operatorname{cl}(C)$. Therefore, $\operatorname{cl}\delta_C(x) = \infty\ \forall x \notin \operatorname{cl}(C)$.

Since $\operatorname{cl}\delta_C(x) = \delta_C(x) = 0\ \forall x \in \operatorname{ri}(C)$, what is left to show is that $\operatorname{cl}\delta_C(y) = 0$ $\forall y \in \operatorname{cl}(C) \setminus \operatorname{ri}(C)$. Fix some $x \in \operatorname{ri}(C)$, and for all $y \in \operatorname{cl}(C) \setminus \operatorname{ri}(C)$ and $\alpha \in (0,1)$, we have $\delta_C\big(y + \alpha(x-y)\big) = 0$ since by Line Segment Principle [COT

Prop. 1.3.1, P. 24], $y + \alpha(x - y) \in \operatorname{ri}(C)$. Thus, by [COT Prop. 1.3.15 (b), P. 40], for all $y \in \operatorname{cl}(C) \setminus \operatorname{ri}(C)$, it holds that

$$\operatorname{cl} \delta_C(y) = \lim_{\alpha \downarrow 0} \delta_C\big(y + \alpha(x - y)\big) = 0.$$

□

The above proof has relied on the set C being convex. In fact, a more general result also hold, which does not rely on convexity, as we will show next.

Lemma (2, P. 89)**.** *Let $X \subset \mathbb{R}^n$ be a nonempty set and δ_X be its indicator function. Then the closure of δ_X, viz., $\operatorname{cl} \delta_X$, is equal to $\delta_{\operatorname{cl}(X)}$.*

Proof. We first note that, in view of [COT Prop. 1.1.2, P. 10], the function $\delta_{\operatorname{cl}(X)}$ is closed as all its level sets are either $\operatorname{cl}(X)$ or $\emptyset$, being closed either way. Then we see that the $\delta_{\operatorname{cl}(X)}$ is majored by δ_X since $\delta_{\operatorname{cl}(X)}(x) = \delta_X(x)$ for $x \in X$ and $\delta_{\operatorname{cl}(X)}(x) \leq \delta_X(x)$ otherwise. So $\delta_{\operatorname{cl}(X)}$ is a closed function majored by δ_X.

Next, denote as V_0 is 0-level set of $\operatorname{cl} \delta_X$. Since $\operatorname{cl} \delta_X \leq \delta_X$, then $X \subset V_0$. By [COT Prop. 1.1.2, P. 10], V_0 is closed. Therefore, we have $\operatorname{cl}(X) \subset V_0$, namely $\operatorname{cl} \delta_X(x) \leq 0$ for all $x \in \operatorname{cl}(X)$. By [COT Prop. 1.3.14 (a), P. 39], $\operatorname{cl} \delta_X(x) \geq \delta_{\operatorname{cl}(X)}$ for all x. Thus, $\operatorname{cl} \delta_X(x) = 0$ for all $x \in \operatorname{cl}(X)$ and $\operatorname{cl} \delta_X(x) = \infty$ for all $x \in \mathbb{R}^n \setminus \operatorname{cl}(X)$. □

P. 89 Given a set X, its indicator function δ_X is closed if and only if X is closed. This can be seen by applying [COT Prop. 1.1.2, P. 10].

Chapter 3

P. 121 A direct consequence of $X^* \neq \emptyset$ is that $f^* \in \mathbb{R}$, since $f^* = f(x)$ for all $x \in X^*$ and $X^* \subset X \cap \operatorname{dom}(f) \neq \emptyset$.

P. 124

Lemma (P. 124)**.** *Let $\{a_n\} \subset \mathbb{R}$ be a convergent real sequence with a limit $\lim_{n\to\infty} a_n$ in $\mathbb{R}^*$. Then for any $\alpha \in \mathbb{R}$, the sequence $\{\alpha a_n\} \subset \mathbb{R}$ is convergent with its limit given by*

$$\lim_{n\to\infty} \alpha a_n = \alpha \lim_{n\to\infty} a_n.$$

Proof. We neglect the case where $\lim_{n\to\infty} a_n \in \mathbb{R}$. We consider the case $\lim_{n\to\infty} a_n = \infty$, while the other case is entirely similar.

Given $\lim_{n\to\infty} a_n = \infty$, if $\alpha = 0$, by arithmetic rule, we have $\alpha \lim_{n\to\infty} a_n = 0$, while $\alpha a_n = 0$ for all n therefore $\{\alpha a_n\}$ convergent and the result holds. If $\alpha > 0$, we have $\alpha \lim_{n\to\infty} a_n = \infty$, and it's clear that the sequence $\{\alpha a_n\}$ has limit ∞. The case where $\alpha < 0$ is neglected. □

Denote as $\bar{a}_k = F(\bar{x}, \bar{z}_k)$ and $\tilde{a}_k = F(\tilde{x}, \tilde{z}_k)$. Since $\bar{a}_k \to f(\bar{x}) < \infty$ and $\tilde{a}_k \to f(\tilde{x}) < \infty$, and F takes values in $(-\infty, \infty]$, it is then without loss of generality to assume that $\{\bar{a}_k\}$, $\{\tilde{a}_k\} \subset \mathbb{R}$. For any $\alpha \in (0,1)$, denote as $\bar{b}_k = \alpha\bar{a}_k$, $\tilde{b}_k = (1-\alpha)\tilde{a}_k$. It is clear by [COT Note Lemma, P. 124] that $\{\bar{b}_k\}$, $\{\tilde{b}_k\} \subset \mathbb{R}$ are both convergent and has limits in $[-\infty, \infty)$. Then by [COT Note Lemma 1, P. 13] and its followup discussion, we have

$$\lim_{k\to\infty} (\bar{b}_k + \tilde{b}_k) = \lim_{k\to\infty} \bar{b}_k + \lim_{k\to\infty} \tilde{b}_k. \tag{49}$$

In addition, by [COT Note Lemma, P. 124], we have $\lim_{k\to\infty} \bar{b}_k = \alpha \lim_{k\to\infty} \bar{a}_k$, $\lim_{k\to\infty} \tilde{b}_k = (1-\alpha) \lim_{k\to\infty} \tilde{a}_k$. In summary, what we've shown above is that for any $\alpha \in (0,1)$,

$$\begin{aligned}\lim_{k\to\infty} \big(\alpha F(\bar{x}, \bar{z}_k) + (1-\alpha) F(\tilde{x}, \tilde{z}_k)\big) =& \alpha \lim_{k\to\infty} F(\bar{x}, \bar{z}_k) + (1-\alpha) \lim_{k\to\infty} F(\tilde{x}, \tilde{z}_k) \\ =& \alpha f(\bar{x}) + (1-\alpha) f(\tilde{x}),\end{aligned}$$

therefore the inequality

$$f\big(\alpha\bar{x} + (1-\alpha)\tilde{x}\big) \le \lim_{k\to\infty} \big(\alpha F(\bar{x}, \bar{z}_k) + (1-\alpha) F(\tilde{x}, \tilde{z}_k)\big) = \alpha f(\bar{x}) + (1-\alpha) f(\tilde{x})$$

follows. For the case where $\alpha = 0$ or 1 the above inequality also holds since $0 \cdot x = 0$ for all $x \in \mathbb{R}^*$.

P. 125

Lemma (P. 125)**.** *Let $F : \mathbb{R}^{n+m} \to (-\infty, \infty]$ be a closed proper convex function. Then for any $\bar{x} \in \mathbb{R}^n$ such that $F(\bar{x}, z) < \infty$ for some $z \in \mathbb{R}^m$, the function $g_{\bar{x}} : \mathbb{R}^m \to [-\infty, \infty]$ given by $g_{\bar{x}}(z) = F(\bar{x}, z)$ is a closd proper convex function.*

Proof. Since by assumption, for the given $\bar{x}$, $F(\bar{x}, z) < \infty$ for some $z \in \mathbb{R}^m$, then $g_{\bar{x}}(z) < \infty$ for some z. In addition, since F is proper, then $g_{\bar{x}}(z) = F(\bar{x}, z) > -\infty$ for all z. Therefore, $g_{\bar{x}}$ is proper.

Next, we will show that $g_{\bar{x}}$ is closed and convex by looking at its epigraph. We denote as $\bar{X}$ the set $\{(x, z, w) \mid x = \bar{x}\}$, and introduce the projection $\bar{P}$ given as

$$\bar{P} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where I is a $m \times m$ identity matrix. Then we look at the set $\bar{P}\big(\mathrm{epi}(F) \cap \bar{X}\big)$. Due to the assumption that $F(\bar{x}, z) < \infty$ for some $z \in \mathbb{R}^m$, $\mathrm{epi}(F) \cap \bar{X} \neq \emptyset$. Since $\mathrm{epi}(F)$ is convex, then $\bar{P}\big(\mathrm{epi}(F) \cap \bar{X}\big)$ is convex. Since $R_{\mathrm{epi}(F) \cap \bar{X}} = R_{\mathrm{epi}(F)} \cap R_{\bar{X}}$ as $\mathrm{epi}(F) \cap \bar{X} \neq \emptyset$, and $R_{\bar{X}} = \{(0, d_z, d_w) \mid (d_z, d_w) \in \mathbb{R}^{m+1}\}$, we see that $N(\bar{P}) \cap R_{\mathrm{epi}(F) \cap \bar{X}} = \{(0,0,0)\}$, which implies that $\bar{P}\big(\mathrm{epi}(F) \cap \bar{X}\big)$ is closed.

In the end, we show that $\mathrm{epi}(g_{\bar{x}}) = \bar{P}\big(\mathrm{epi}(F) \cap \bar{X}\big)$. If $(z, w) \in \mathrm{epi}(g_{\bar{x}})$, then by the definition of $g_{\bar{x}}$, $F(\bar{x}, z) \le w$, which implies $(\bar{x}, z, w) \in \mathrm{epi}(F) \cap \bar{X}$, resulting in $(z, w) \in \bar{P}\big(\mathrm{epi}(F) \cap \bar{X}\big)$. Conversely, if $(z, w) \in \bar{P}\big(\mathrm{epi}(F) \cap \bar{X}\big)$, then $(\bar{x}, z, w) \in \mathrm{epi}(F) \cap \bar{X}$, which implies $F(\bar{x}, z) \le w$. By the definition of $g_{\bar{x}}$, we have $(z, w) \in \mathrm{epi}(g_{\bar{x}})$. Therefore, $g_{\bar{x}}$ is a closed proper convex function. ∎

Appendix A

P. 228 For $f : X \to Y$, and $U \subset X$, $V \subset Y$, our orthodox definition of $f(U)$ and $f^{-1}(V)$ are given as

$$\begin{aligned} f^{-1}(V) =& \{x \in X \mid f(x) \in V\}, \\ f(U) =& \{y \in Y \mid \exists x \in U, \, y = f(x)\}. \end{aligned}$$

P. 229

Lemma (1, P. 229)**.** *Given an affine set $A = x + S$ where S is a subspace, for every $\bar{x} \in A$, it holds that $A = \bar{x} + S$.*

Proof. Note that $\bar{x} \in x + S$, so we have $\bar{x} = x + \bar{s}$ where $\bar{s} \in S$. Therefore, for every $x' = x + s'$, it holds that $x' = x + \bar{s} + s' - \bar{s}$ where $s' - \bar{s} \in S$. Therefore, $x' \in \bar{x} + S$, which gives $x + S \subset \bar{x} + S$. The reverse inclusion uses the same arguments. □

Lemma (2, P. 229)**.** *Given an affine set $A = x + S$ where S is a subspace, if $a_1, \ldots, a_m \in A$, then for scalars $\alpha_1, \ldots, \alpha_m$ such that $\sum_{i=1}^m \alpha_i = 1$, it holds that $\sum_{i=1}^m \alpha_i a_i \in A$.*

Proof. Since $a_i \in A$, then we have $a_i = x + s_i$ where $s_i \in S$ for all $i = 1, \ldots, m$. Then we have $\sum_{i=1}^m \alpha_i a_i = \sum_{i=1}^m \alpha_i (x + s_i) = x + \sum_{i=1}^m \alpha_i s_i \in A$. □

Lemma (3, P. 229)**.** *Given an affine set $A = x + S$ where S is a subspace, if $x \in S$, then $A = S$.*

Proof. Since $x \in S$, then $-x \in S$, which implies $0 = x - x \in A$. By [COT Note Lemma 1, P. 229], $A = 0 + S = S$. □

Lemma (4, P. 229)**.** *Let A be a nonempty set such that for all $a_1, a_2 \in A$, $\alpha a_1 + (1 - \alpha) a_2 \in A$ for all $\alpha \in \mathbb{R}$. Then A is an affine set.*

Proof. Since A is nonempty, then there exists some $\bar{a} \in A$. Then $A = \bar{a} + A - \bar{a}$. Denote as $\bar{S}$ the set $A - \bar{a}$. We will show that $\bar{S}$ is a subspace. For all $x, y \in \bar{x}$, we have $x + \bar{a}, y + \bar{a} \in A$. For any $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} \alpha x + \beta y =& \alpha(x + \bar{a} - \bar{a}) + \beta(y + \bar{a} - \bar{a}) \\ =& \alpha(x + \bar{a}) + \beta(y + \bar{a}) - (\alpha + \beta)\bar{a} \end{aligned}$$

From the property of A, it is easy to show that for all $a_1, a_2, a_3 \in A$, $\alpha a_1 + \beta a_2 + (1 - \alpha - \beta) a_3 \in A$ for all $\alpha, \beta \in \mathbb{R}$. Then we see that $\alpha x + \beta y + \bar{a} = \alpha(x + \bar{a}) + \beta(y + \bar{a}) + (1 - \alpha - \beta)\bar{a} \in A$, which implies $\alpha x + \beta y \in \bar{S}$. Therefore, $\bar{S}$ is a subspace. Then $A = \bar{a} + \bar{S}$ is an affine set. □