January 1, 2026

Chapter 1

P. 2 Suppose that $h(x)$ is nonlinear. For a given x, there may not exist a vector d such that $h(x + \alpha d) = 0$ for all $\alpha > 0$ that are sufficiently small.

P. 2 Let us denote by S the set $\{x \mid h(x) = 0\}$. First, we have,

$$\lim_{c_k \to \infty} f(x) + \frac{1}{2} c_k |h(x)|^2 = \begin{cases} f(x) & \text{if } x \in S, \\ \infty & \text{otherwise.} \end{cases}$$

If $S = \emptyset$, we have $\inf_{x \in S} f(x) = \infty$, and

$$\lim_{c_k \to \infty} f(x) + \frac{1}{2} c_k |h(x)|^2 = \infty$$

for all $x \in \Re^n$ so that its infimum is also ∞. Otherwise, if S is nonempty, then

$$\inf_{x \in \Re^n} \lim_{c_k \to \infty} f(x) + \frac{1}{2} c_k |h(x)|^2 = \inf_{x \in S} \lim_{c_k \to \infty} f(x) + \frac{1}{2} c_k |h(x)|^2 = \inf_{x \in S} f(x).$$

P. 8 Let us denote by a_{ij}^k the ijth element of matrix A_k. From the definition of matrix norm $|A_k|$, it can be seen that $|A_k| \geq |a_{ij}^k|$. Therefore, $\lim_{k \to \infty} |A_k| = 0$ implies that $\lim_{k \to \infty} a_{ij}^k = 0$.

P.8

Lemma (1, P. 8)**.** *Let $\{r_k\}$ be a sequence on the real line. There exists a subsequence $\{r_k\}_K$ that is monotone.*

Proof. Consider the index set defined as $\overline{K} = \{k \mid r_n \geq r_k \text{ for all } n \geq k\}$. If the set $\overline{K}$ is infinite, by setting $K = \overline{K}$, we obtain a sequence $\{r_k\}_K$ that is monotonically nondecreasing.

Suppose that the set $\overline{K}$ is finite. Let k_1 be some index that does not belong to $\overline{K}$ and $k_1 > \max \overline{K}$. Then there exists some $k_2 > k_1$ such that $r_{k_1} > r_{k_2}$. Since $k_2 > k_1$, then $k_2 \notin \overline{K}$. Therefore, there exists some $k_3 > k_2$ such that $r_{k_2} > r_{k_3}$. Continue in this way, we construct the set $K = \{k_1, k_2, \dots\}$ and the corresponding subsequence is monotonically decreasing. Q.E.D.

The following result is known as the Bolzano-Weierstrass theorem.

Lemma (2, P. 8)**.** *Let $\{r_k\}$ be a bounded sequence on the real line. There exists a subsequence $\{r_k\}_K$ that is convergent.*

Proof. According to [Lemma 1, P.8], $\{r_k\}$ has a monotone subsequence. We assume first that $\{r_k\}_K$ is a nondecreasing subsequence. Then one can see that its limit is $\sup_{k\in K}\{r_k\}$. The case where $\{r_k\}$ has a nonincreasing subsequence can be show similarly. Q.E.D.

Lemma (3, P. 8)**.** *Let $\{s_k\} \subset \Re^n$ be a bounded sequence. There exists a subsequence $\{s_k\}_K$ that is convergent.*

Proof. Let us denote by s_k^i the ith component of s_k. Since $\{s_k\}$ is bounded, the sequences $\{s_k^i\}$, $i = 1, 2, \ldots, n$, are all bounded. Then there exists a subsequence $\{s_k^1\}_{K_1}$ that is bounded according to [Lemma 2, P.8]. Consider now the sequence $\{s_k^2\}_{K_1}$. Similarly, [Lemma 2, P.8] implies that there exists some $K_2 \subset K_1$ such that $\{s_k^2\}_{K_2}$ is convergent. Repeating this n steps, and we get some set K_n such that $\{s_k^n\}_{K_n}$ is convergent. Moreover, since any subsequence of a convergent sequence is convergent, we have $\{s_k^i\}_{K_n}$, $i = 1, \ldots, n-1$, are convergent. As a result, $\{s_k\}_{K_n}$ is convergent. Q.E.D.

P. 8 Suppose that $\{r_k\}$ is unbounded below. Its limit superior can be either finite or $-\infty$. To see this, consider $r_k = -k$ when k is even and $r_k = 1/k$ when k is odd. In this case, we have $\limsup_{k\to\infty} = 0$.

P. 9 Let us denote by S the set $\cap_{k=0}^{\infty} S_k$. To see that S is nonempty, consider a sequence $\{s_k\}$ where $s_k \in S_k$, as S_k is nonempty for all k. Since the sets $S_k \subset \Re^n$ are compact, we may use certain canonical way to construct $\{s_k\}$ so that axiom of countable choice is not needed.[1] Then using the construction in the proof of [Lemma 3, P. 8], we obtain a subsequence $\{s_k\}_K$ that is convergent. Denote by $\bar{s}$ the limit of $\{s_k\}_K$. Since S_k, $k = 0, 1, \ldots$, are closed, then $\bar{s} \in S_k$ for all k. Therefore, $\bar{s} \in S$.

[1] Suppose that X belongs to $\Re^n$ and is compact. We show that there is a canonical way to select a vector $x \in X$. Since the function $f_0(x) = |x|$ is continuous on x and X is compact, we have that $\inf_{x\in X} |x|$ is finite. Let $\{x_k\}$ be a sequence such that $f_0(x_k)$ converges to $\inf_{x\in X} |x|$. Such a sequence exists due to the definition of infimum. Since the sequence is also bounded, there exists a subsequence $\{x_k\}_K$ that is convergent. The limit of $\{x_k\}_K$, denoted by x^*, belongs to X due to X being compact. Given that f_0 is continuous and $\lim_{k\in K} f_0(x_k) = \inf_{x\in X} |x|$, we have $f_0(x^*) = \inf_{x\in X} |x|$. As a result, the set $X_0 = \arg\min_{x\in X} |x|$ is nonempty and contains at least x^*. Moreover, it is closed since every sequence that belongs to X_0 has its limit in X_0 (due to X being compact and f_0 being continuous) and is also bounded. Next, consider the function $f_1(x) = x^1$ where x^1 denotes the first element of x. Using entirely identical arguments with f_1 in place of f and X_0 in place of X, we obtain a compact set X_1.

After proceeding similarly, we construct the sequence of sets $X_1, X_2, \ldots, X_n$. We surely have X_n as a singleton. To see this, we note that by construction, X_{n-1} contains vectors x with their first $n-1$ elements being equal. Then the vector with the minimum last element must be unique. We can use the unique element in X_n as our canonical choice.

To see that S is closed, let $\{s_k\}$ be some convergent sequence in S. Then $\{s_k\} \subset S_k$ for all k. Let $\bar{s}$ be the limit of $\{s_k\}$. Since S_k, $k = 0, 1, \ldots,$ are closed, then $\bar{s} \in S_k$ for all k. Therefore, $\bar{s} \in S$.

P. 9 We say a function $f : S_1 \mapsto S_2$ is continuous at $x \in S_1$ if $f(x_k) \to f(x)$ whenever $\{x_k\} \subset S$ and $s_k \to x$; see [NLP 3rd, P. 754].

P. 9 For a function $f : X \mapsto \Re$, where $X \subset \Re^n$ is an open set, its partial derivatives of order 0 is simply the function itself. As a result, if f is continuous over X, we write $f \in C^0$ over X.

P. 10 For the definitions of $\nabla^2_{xx} f(x, y)$, $\nabla^2_{xy} f(x, y)$, $\nabla^2_{yy} f(x, y)$, see [NLP P. 767]. If $f : X \mapsto \Re$ is a real-valued function of (x, y), where $x = (x_1, \ldots, x_n) \in \Re^n$, $y = (y_1, \ldots, y_r) \in \Re^r$, and $X \subset \Re^{n+r}$ is an open set, then the notations $\nabla_x f(x, y)$, $\nabla_y f(x, y)$, and the correct notations $\nabla^2_{xx} f(x, y)$, $\nabla^2_{xy} f(x, y)$, $\nabla^2_{yy} f(x, y)$ introduced here remain valid. Similarly, if $f : X \mapsto \Re^m$ where $X \subset \Re^{n+r}$ is an open set, the notation $\nabla_x f(x, y)$ and $\nabla_y f(x, y)$ introduced here remain valid.

P. 11 It is implied implicitly in Implicit Function Theorem 1 that for all $x \in S(\bar{x}; \epsilon)$, $\big(x, \phi(x)\big) \in S$.

P. 12 We prove Implicit Function Theorem 2 by using Implicit Function Theorem 1. We make use of the following lemma.

Lemma (1, P. 12)**.** *Let $X \subset \Re^n$ be a compact set and $Y \subset \Re^n$ be an open set such that $X \subset Y$. There exists some $\epsilon > 0$ such that $S(X; \epsilon) \subset Y$.*

Proof. When $Y = \Re^n$, the conclusion holds trivially. Otherwise, denote by Z the complement of Y in $\Re^n$, which is nonempty and closed. Consider the scalar δ defined as

$$\delta = \inf_{x \in X, z \in Z} |x - z|.$$

We will show that $\delta > 0$ by contradiction. Suppose $\delta = 0$, then there exists some sequence $\{x_k\} \subset X$ and $\{z_k\} \subset Z$ such that $|x_k - z_k| \to 0$. Since the set X is compact, there exists a convergent subsequence $\{x_k\}_K$. Denote by x^* the limit of $\{x_k\}_K$ and we have $x^* \in X$. Moreover, x^* is also the limit of the subsequence of $\{z_k\}_K$ since $|z_k - x^*| \leq |z_k - x_k| + |x_k - x^*|$. As Z is a closed set, this implies that $x^* \in Z$. This leads to a contradiction as $x^* \in X \subset Y$ and Z is the complement of Y. Therefore, we have $\delta > 0$.

Define $\epsilon = \delta/2$. We will show that $S(X; \epsilon) \subset Y$. If $y \in S(X; \epsilon)$, then there exists some $x \in X$ such that $|x - y| < \epsilon < \delta$, which means that $y \notin Z$ by the definition of δ. Q.E.D.

Denote by $\bar{x}$ some arbitrary point in $\overline{X}$. Applying Implicit Function Theorem 1, there exists some $\epsilon_{\bar{x}} > 0$ and $\delta_{\bar{x}} > 0$, and a function $\phi_{\bar{x}} : S(\bar{x}; \epsilon_{\bar{x}}) \mapsto S(\bar{y}; \delta_{\bar{x}})$ such that $h[x, \phi_{\bar{x}}(x)] = 0$ for all $x \in S(\bar{x}; \epsilon_{\bar{x}})$ and $\phi_{\bar{x}}(\bar{x}) = \bar{y}$. Since $h(x, \bar{y}) = 0$

for all $x \in \bar{X}$, we have that $\phi_{\bar{x}}(x) = \bar{y}$ for all $x \in S(\bar{x};\epsilon_{\bar{x}}) \cap \overline{X}$ by the uniqueness of the function $\phi_{\bar{x}}(x)$. Moreover, if $p \geq 1$, we have for all $x \in S(\bar{x};\epsilon_{\bar{x}})$

$$\nabla \phi_{\bar{x}}(x) = -\nabla_x h[x, \phi_{\bar{x}}(x)] \big[\nabla_y h[x, \phi_{\bar{x}}(x)]\big]^{-1}.$$

Next, we note that the set $\cup_{\bar{x}\in\overline{X}} S(\bar{x};\epsilon_{\bar{x}})$ is an open set and $\overline{X} \subset \cup_{\bar{x}\in\overline{X}} S(\bar{x};\epsilon_{\bar{x}})$. Given that $\overline{X}$ is compact, there exists some vectors $\overline{x}_i$, $i = 1, 2, \ldots, m$, such that $\overline{X} \subset \cup_{i=1}^m S(\bar{x}_i;\epsilon_{\bar{x}_i})$. Let $\bar{x} \in \cup_{i=1}^m S(\bar{x}_i;\epsilon_{\bar{x}_i})$. We will show that for the values $\phi_{\bar{x}_i}(\bar{x})$ agree for all i in the set $I_{\bar{x}} = \{i \,|\, \bar{x} \in S(\bar{x}_i;\epsilon_{\bar{x}_i})\}$. Suppose that $i^* \in \arg\max_{i\in I_{\bar{x}}} \delta_{\bar{x}_i}$ so that $S(\bar{y};\delta_{\bar{x}_i}) \subset S(\bar{y};\delta_{\bar{x}_{i^*}})$ for all $i \in I_{\bar{x}}$. In other words, the values of $\phi_{\bar{x}_i}(\bar{x})$, $i = 1, 2, \ldots, m$, are all within $S(\bar{y};\delta_{\bar{x}_{i^*}})$. Applying the uniqueness property of $\phi_{\bar{x}_{i^*}}$, we have that $\phi_{\bar{x}_i}(\bar{x})$ agree for all i and we can define a function ϕ that maps $\cup_{i=1}^m S(\bar{x}_i;\epsilon_{\bar{x}_i})$ to $\cup_{i=1}^m S(\bar{y};\delta_{\bar{x}_i})$ by setting its value at $\bar{x}$ equal to $\phi_{\bar{x}_i}(\bar{x})$ for any $i \in I_{\bar{x}}$. Applying [Lemma 1, P. 12], there exists some $\epsilon > 0$ such that $S(\overline{X};\epsilon) \subset \cup_{i=1}^m S(\bar{x}_i;\epsilon_{\bar{x}_i})$. By setting $\delta = \max_{i=1,\ldots,m} \delta_i$, we complete the proof.

P. 13 Consider the sequence $q(\beta+1/k)^k$. We will show that it converges linearly with convergence rate β by verification. Let $\bar{q} > 0$ and $\bar{\beta} \in (\beta, 1)$. Then there exists some $\tilde{k}$ such that $\beta + 1/\tilde{k} < \bar{\beta}$. Moreover, there exists some $\bar{k} \geq \tilde{k}$ such that $(\beta + 1/\tilde{k})^{\bar{k}}/\bar{\beta}^{\bar{k}} \leq \bar{q}/q$. As a result, for all $k \geq \bar{k}$, we have

$$\begin{aligned} q(\beta + 1/k)^k \leq q(\beta + 1/\tilde{k})^k = q(\beta + 1/\tilde{k})^{k-\bar{k}}[(\beta + 1/\tilde{k})^{\bar{k}}/\bar{\beta}^{\bar{k}}]\bar{\beta}^{\bar{k}} \\ \leq q(\beta + 1/\tilde{k})^{k-\bar{k}}[\bar{q}/q]\bar{\beta}^{\bar{k}} \leq \bar{q}\bar{\beta}^k. \end{aligned}$$

The verification for the sequence converging at most at ratio β is straightforward.

The convergence speed of the sequence $q\beta^{k+(1/k)}$ can be verified by making use of the following result.

Lemma (1, P. 13)**.** *Let $a > 0$ be some scalar and consider the function $f : (0,\infty) \mapsto \Re$ defined as $f(x) = a^{1/x}$. There holds $\lim_{x\to\infty} f(x) = 1$. Moreover, if $a < 1$, we have $f(x) < 1$ for all x, and if $a > 1$, we have $f(x) > 1$.*

Proof. Note that $a^{(1/x)} = \left(e^{(\ln a)}\right)^{(1/x)} = e^{(\ln a)/x}$. Then the conclusion follows. Q.E.D.

In addition, for the function $f : (0,\infty) \mapsto \Re$ defined as $f(x) = 0^x$, we have $f(x) = 0$ for all $x \in (0,\infty)$.

P. 15 In what follows, we either fill in some details or provide alternative arguments for the proofs.

(a) Let $q > 0$, $\bar{\beta} \in (\beta, 1)$, and $\tilde{\beta} \in (\beta, \bar{\beta})$. Since $\limsup_{k\to\infty} e_k^{1/k} \leq \beta$, there exists some $\tilde{k}$ such that $\sup_{k\geq\tilde{k}} e_k^{1/k} \leq \tilde{\beta}$, which means that $e_k \leq \tilde{\beta}^k$ for all $k \geq \tilde{k}$. Since $\tilde{\beta} \leq \bar{\beta}$, there exists some $\bar{k} \geq \tilde{k}$ such that $(\tilde{\beta}/\bar{\beta})^{\bar{k}} \leq q$. As a result, for all

$k \geq \bar{k}$, $\tilde{\beta}^k = \tilde{\beta}^{k-\bar{k}}(\tilde{\beta}/\bar{\beta})^{\bar{k}}\bar{\beta}^{\bar{k}} \leq q\beta^{k-\bar{k}}\bar{\beta}^{\bar{k}} \leq q\bar{\beta}^k$. This shows that the sequence e_k converges at least linearly with ratio β. Conversely, since for every $\bar{\beta} \in (\beta, 1)$, there exists some $\bar{k}$ such that $e_k \leq \bar{\beta}^k$ for all $k \geq \bar{k}$, or equivalently $e_k^{1/k} \leq \bar{\beta}$. In other words, we have $\sup_{k \geq \bar{k}} e_k^{1/k} \leq \bar{\beta}$. Using the fact that $\sup_{k \geq \bar{k}} e_k^{1/k}$ is decreasing with $\bar{k}$, we have $\limsup_{k\to\infty} e_k^{1/k} \leq \bar{\beta}$. Since the inequality holds for all $\bar{\beta} \in (\beta, 1)$, we have $\limsup_{k\to\infty} e_k^{1/k} \leq \inf(\beta, 1) = \beta$.

(b) The proof presented in [COLMM] uses the fact that $\lim_{k\to\infty} q^{1/k} = 1$; see [Lemma 1, P. 13]. Next, we provide an alternative proof. Suppose that for some $\tilde{k}$, we have $e_k \leq q\beta^k$ for all $k \geq \tilde{k}$. Let $\bar{q} > 0$ and $\bar{\beta} \in (\beta, 1)$. There exists some $\bar{k} \geq \tilde{k}$ such that $(\beta/\bar{\beta})^{\bar{k}} \leq \bar{q}/q$. Using the derivation as in part (a), we have $q\beta^k \leq \bar{q}\bar{\beta}^k$ for all $k \geq \bar{k}$.

(c) The proof presented in [COLMM] uses the fact that $\lim_{k\to\infty} \bar{\beta}_2^{1/k} = 1$; see [Lemma 1, P. 13]. Alternatively, let $\bar{q} > 0$, and $\bar{\beta}_2 \in (\beta_2, 1)$. Consider $\tilde{\beta}_2 \in (\beta_2, \bar{\beta}_2)$. There exists some $\tilde{k}$ such that $e_{k+1}/e_k \leq \tilde{\beta}_2$ for all $k \geq \tilde{k}$. As a result, we have $e_k \leq e_{\tilde{k}}\tilde{\beta}^{k-\tilde{k}} = q\tilde{\beta}^k$ for all $k \geq \tilde{k}$, where $q = e_{\tilde{k}}/\tilde{\beta}^{\tilde{k}}$. Using part (b), we have that e_k converges faster than $\bar{q}\bar{\beta}_2^k$.

P. 16 (a) Consider $q > 0$, $\beta \in (0, 1)$, and $\bar{p} \in (1, p)$. Let $\bar{\beta} \in (0, \beta)$. Since $\lim_{k\to\infty} e_k^{1/\bar{p}^k} = 0$, there exists some $\tilde{k}$ such that $e_k^{1/\bar{p}^k} \leq \bar{\beta}$ for all $k \geq \tilde{k}$, or equivalently $e_k \leq \bar{\beta}^{\bar{p}^k}$ for all $k \geq \tilde{k}$. Since $\tilde{\beta} < \beta$, then there exists some $\bar{k} \geq \tilde{k}$ such that $(\tilde{\beta}/\beta)^{\bar{p}^k} < q$ for all $k \geq \bar{k}$. Then we have $e_k \leq \bar{\beta}^{\bar{p}^k} = (\bar{\beta}/\beta)^{\bar{p}^k}\beta^{\bar{p}^k} \leq q\beta^{\bar{p}^k}$ for all $k \geq \bar{k}$. Conversely, fix some $\bar{p}$. Since for every $\beta \in (0, 1)$, there exists some $\bar{k}$ such that $e_k \leq \beta^{\bar{p}^k}$ for all $k \geq \bar{k}$. Equivalently, we have $e_k^{1/\bar{p}^k} \leq \beta$ for all $k \geq \bar{k}$. Since β can be arbitrarily small and $e_k^{1/\bar{p}^k} \geq 0$, we have $\lim_{k\to\infty} e_k^{1/\bar{p}^k} = 0$.

For the statement regarding the convergence at most superlinearly with order $p > 1$, the proof is similar, except that in the only if part, fixing $\bar{p}$, we also need to show that there exists some $\bar{k}$ such that $e_k^{1/\bar{p}^k} \leq 1$ for all $k \geq \bar{k}$. Indeed, since $e_k \to 0$, there exists some $\bar{k}$ such that $e_k \leq 1$ for all $k \geq \bar{k}$. Then we have $e_k^{1/\bar{p}^k} \leq 1^{1/\bar{p}^k} = 1$ for all $k \geq \bar{k}$.

(b) Suppose there exists some $\bar{k}$ such that $e_k \leq q\beta^{p^k}$ for all $k \geq \bar{k}$. For every $\bar{p} \in (1, p)$, we have $e_k^{1/\bar{p}^k} \leq q^{1/\bar{p}^k}\beta^{(p/\bar{p})^k}$ for all $k \geq \bar{k}$. Taking limit superior on both sides, we have $\limsup_{k\to\infty} e_k^{1/\bar{p}^k} \leq \lim_{k\to\infty} q^{1/\bar{p}^k}\beta^{(p/\bar{p})^k} = 0$. In view of the fact that $e_k^{1/\bar{p}^k} \geq 0$ for all k, this implies that $\lim_{k\to\infty} = e_k^{1/\bar{p}^k} = 0$. Applying part (a) we get the desired result. Similarly we prove the result concerning the convergence at most superlinearly with order p.

(c) We prove the part concerning the convergence at least superlinearly with order p. Since $\limsup_{k\to\infty}(e_{k+1}/e_k^p) < \infty$, there exists some $\tilde{k}$ and $c > 0$ such that for all $k \geq \bar{k}$, we have $e_{k+1} \leq ce_k^p$ for all $k \geq \tilde{k}$. This implies that

$e_{\bar{k}+m} \le c^{(\sum_{i=0}^{m-1} p^i)} e_{\bar{k}}^{p^m}$ for all $\bar{k} \ge \tilde{k}$ and $m \ge 1$. Therefore, we obtain the inequality

$$e_{\bar{k}+m}^{1/\bar{p}^{\bar{k}+m}} \le c^{(\sum_{i=0}^{m-1} p^i)/\bar{p}^{\bar{k}+m}} e_{\bar{k}}^{p^m/\bar{p}^{\bar{k}+m}}. \tag{1}$$

Focusing on the exponent of c, we have

$$\left(\sum_{i=0}^{m-1} p^i\right) \Big/ \bar{p}^{\bar{k}+m} = \frac{p^m - 1}{(p-1)\bar{p}^{\bar{k}+m}} = \frac{1}{(p-1)\bar{p}^{\bar{k}}} \big((p/\bar{p})^m - 1/\bar{p}^m\big).$$

There holds that

$$\begin{aligned} & c^{(\sum_{i=0}^{m-1} p^i)/\bar{p}^{\bar{k}+m}} e_{\bar{k}}^{p^m/\bar{p}^{\bar{k}+m}} \\ =& \left(c^{1/(p-1)}\right)^{\left(1/\bar{p}^{\bar{k}}\right)(p/\bar{p})^m} \left(c^{1/(p-1)}\right)^{\left(-1/\bar{p}^{\bar{k}+m}\right)} e_{\bar{k}}^{\left(1/\bar{p}^{\bar{k}}\right)(p/\bar{p})^m} \\ =& \left(c^{1/(p-1)} e_{\bar{k}}\right)^{\left(1/\bar{p}^{\bar{k}}\right)(p/\bar{p})^m} \left(c^{1/(p-1)}\right)^{\left(-1/\bar{p}^{\bar{k}+m}\right)}. \end{aligned}$$

Since $e_k \to 0$, we can choose $\bar{k}$ sufficiently large such that $c^{1/(p-1)} e_{\bar{k}} < 1$. In view of $\bar{p} < p$, we have that

$$\lim_{m\to\infty} \left(c^{1/(p-1)} e_{\bar{k}}\right)^{\left(1/\bar{p}^{\bar{k}}\right)(p/\bar{p})^m} = 0, \quad \lim_{m\to\infty} \left(c^{1/(p-1)}\right)^{\left(-1/\bar{p}^{\bar{k}+m}\right)} = 1.$$

Therefore, taking the limit superior with respect to m on both sides of Eq. (1), we have $\limsup_{k\to\infty} e_k^{1/\bar{p}^k} \le 0$. Also we have $e_k^{1/\bar{p}^k} \ge 0$ for all k, therefore we obtain $e_k^{1/\bar{p}^k} \to 0$. Applying part (a) completes the proof. Similarly we prove the result concerning the convergence at most superlinearly with order p.

P. 17 In particular, we have $a_{11} > 0$.

P. 17 Let A be a positive definite matrix. It can be written as $A = LL'$, where L is a unique lower triangular matrix whose diagonal elements are positive.

P. 18 To see uniqueness, by induction we have L_1 is unique. Suppose that A_{i-1} can be uniquely written as $A_{i-1} = L_{i-1}L'_{i-1}$. Suppose that

$$A = \begin{bmatrix} \tilde{L}_{i-1} & 0 \\ l'_i & \lambda_{ii} \end{bmatrix}.$$

Then we have $\tilde{L}_{i-1}\tilde{L}'_{i-1} = A_{i-1}$, $\tilde{L}_{i-1} l_i = \alpha_i$, and $l'_i l_i + \lambda_{ii}^2 = a_{ii}$. By induction hypothesis, we have $\tilde{L}_{i-1} = L_{i-1}$. Moreover, the vector l_i is uniquely determined by the equations $\tilde{L}_{i-1} l_i = \alpha_i$. The uniqueness of λ_{ii} follows as it is required to be positive.

P. 18 The computation $L_{i-1} l_i = \alpha_i$ requires $i(i-1)/2$ multiplications. For a matrix of dimension n, the total multiplication involved is

$$\frac{1}{2}\sum_{i=2}^{n}(i^2 - i).$$

Using the equation $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$, we have that the total multiplication is approximately $n^3/6$.

P. 20 The proposition here should be interpreted as follows.

Lemma (1, P. 20)**.** *Let $f : \Re^n \mapsto \Re$ be a continuous function. If either*

(a) for every sequence $\{x_k\}$ satisfying $|x_k| \to \infty$, we have $f(x_k) \to \infty$, or

(b) more generally, for some $\alpha \in \Re$, the set $\{x \mid f(x) \leq \alpha\}$ is nonempty and compact,

then there exists a global minimum for (UP).

Proof. Denote $a = \inf_{x \in \Re^n} f(x)$. Clearly we have $a \in [-\infty, \infty)$, and there exists some sequence x_k such that $f(x_k) \to a$.

Suppose condition (a) hold. We show that the sequence $\{x_k\}$ is bounded by contradiction. If $\{x_k\}$ is unbounded, then there exists some subsequence $\{x_k\}_K$ such that $|x_k| \to \infty$ as $k \in K$ and $k \to \infty$. This implies that $\lim_{k \in K, k \to \infty} f(x_k) = \infty$, which is a contradiction. Therefore, the sequence $\{x_k\}$ is bounded. By [Lemma 3, P. 8], there exists a subsequence $\{x_k\}_{\overline{K}}$ that is convergent. Denote by x^* its limit. Then we have $\lim_{k \to \infty} f(x_k) = \lim_{k \in \overline{K}, k \to \infty} f(x_k) = f(x^*) > \infty$. Moreover, $f(x^*) = a$, thus attaining the global minimum.

Next, we show that condition (a) implies condition (b). As is shown, we have $a > \infty$. Let $\alpha > a$ and consider the set $X = \{x \mid f(x) \leq \alpha\}$. It is nonempty since $x^* \in X$. Moreover, it is closed since f is continuous. Suppose that X is unbounded. Then there exists some $\{x_k\}$ such that $|x_k| \to \infty$. However, this implies that $f(x_k) \to \infty$ as per condition (a). Therefore, X must be bounded, thus compact.

Lastly, suppose condition (b) hold. Since the set $X = \{x \mid f(x) \leq \alpha\}$ is nonempty, we have $a \leq \alpha$. If $a = \alpha$, then every element in the set X attains the minimum. Otherwise, there exists some $\bar{k}$ such that $x_k \in X$ for all $k \geq \bar{k}$. Since X is compact, there exists a subsequence $\{x_k\}_{k \geq \bar{k}, k \in K}$ that is convergent. Its limit, denoted by x^*, attains the global minimum. Q.E.D.

Note that there exists some functions that satisfy condition (b) but not (a). For example, consider $f : \Re \mapsto \Re$ defined as

$$f(x) = \begin{cases} -x - 1 & \text{if } x < 0, \\ x - 1 & \text{if } 0 \leq x < 3, \\ \frac{6}{x} & \text{otherwise.} \end{cases}$$

Here condition (b) is satisfied while condition (a) is not. Also, without f being continuous, condition (b) alone is not sufficient to ensure the existence of global minimum. To see this, consider the function $f : \Re \mapsto \Re$ defined as

$$f(x) = \begin{cases} -\ln x & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ \ln x & \text{otherwise.} \end{cases}$$

P. 20

Lemma (2, P. 20)**.** *Let $h : [0, \kappa) \mapsto \Re$ be a continuous function, where κ is a positive real number or ∞. Suppose that for some real numbers $a \in \Re$, we have $h(0) < a$. Then there exists some $\bar{\alpha} \in (0, \kappa)$ such that $h(\alpha) < a$ for all $[0, \bar{\alpha}]$.*

Proof. Suppose there is no such $\bar{\alpha}$. Then for all $1/k > 0$, there exists some $\alpha_k \in (0, 1/k]$ such that $h(\alpha_k) \geq a$. Then we have $\lim_{k\to\infty} h(\alpha_k) \geq a$, which contradicts with the continuity of h at 0. Q.E.D.

Lemma (3, P. 20)**.** *Let $\gamma \in (0, 1)$ and $\nabla f(x)'d < 0$, where $f \in C^1$ over $\Re^n$. There exists an interval $(0, \bar{\alpha}]$ such that every $\alpha \in (0, \bar{\alpha}]$ satisfies*

$$f(x) - f(x + \alpha d) \geq -\gamma\alpha\nabla f(x)'d.$$

Proof. Define $g(\alpha) = f(x + \alpha d)$. Note that $\partial g(\alpha)/\partial\alpha = \nabla f(x + \alpha d)'d$. Define $h(\alpha) = \partial g(\alpha)/\partial\alpha$ for $\alpha \geq 0$, which is continuous. Clearly we have $h(0) < 0$, and, as a result, $h(0) < \gamma h(0)$. By [Lemma 2, P. 20] with $a = \gamma h(0)$ and $\kappa = \infty$, there exists some $\bar{\alpha}$ such that $h(\tilde{\alpha}) < \gamma h(0)$ for all $\tilde{\alpha} \in (0, \bar{\alpha}]$, or equivalently, $\partial g(\tilde{\alpha})/\partial\alpha < \gamma\partial g(0)/\partial\alpha$. Consequently, for every $\tilde{\alpha} \in (0, \bar{\alpha}]$, there holds

$$g(\tilde{\alpha}) = g(0) + \int_0^{\tilde{\alpha}} \frac{\partial g(t)}{\partial\alpha} dt < g(0) + \gamma\frac{\partial g(0)}{\partial\alpha}\tilde{\alpha},$$

which is equivalent to the desired inequality. Q.E.D.

Applying [Lemma 3, P. 20], we have that for given scalars s, β, and σ with $s > 0$, $\beta \in (0, 1)$, and $\sigma \in (0, \frac{1}{2})$, Armijo rule is always well defined. Indeed, by setting $\gamma = \sigma$, we see that the inequality used in Armijo rule is satisfied for sufficiently large m.

P. 21 In Goldstein rule, we must have $\alpha_k > 0$, as otherwise it would imply $f(x_{k+1}) > f(x_k)$.

P 22

Lemma (1, P. 22)**.** *Let $h : [0, \kappa) \mapsto \Re$ be a continuous function, where κ is a positive real number or ∞. Suppose that for some real numbers $a, b \in \Re$ such that $a < b$, we have $h(0) < a$ and $h(\alpha) \notin [a, b]$ for all $\alpha \in [0, \kappa)$. There holds that $\sup_{\alpha\in[0,\kappa)} h(\alpha) \leq a$.*

Proof. Define c as $c = \sup X$, where

$$X = \{\tilde{\alpha} \mid 0 \leq \tilde{\alpha} < \kappa, \text{ and } h(\alpha) < a \text{ for all } \alpha \in [0, \tilde{\alpha}]\}.$$

In view of [Lemma 2, P. 20], the set X is nonempty and contains some $\alpha > 0$. If c is κ, we have $\sup_{\alpha \in [0,\kappa)} h(\alpha) \leq a$. Otherwise, assume $c < \kappa$. As a result, we have $(c + 1/k) \notin X$ and $c + 1/k < \kappa$ for all sufficiently large k. Therefore, there exits some $c_k \in [0, c + 1/k]$ such that $h(c_k) \geq a$. Since $h(\alpha) \notin [a, b]$ for all $\alpha \in [0, \kappa)$, we have that $h(c_k) \geq b$ for all k. Moreover, in view of the definition of c, we have $c_k \geq c$. We can show that the function h is not continuous at c by considering the sequences $\{c - 1/k\}$ and $\{c_k\}$, which is a contradiciton. Therefore, we have $c = \kappa$. Q.E.D.

Define $h(\alpha) = \partial g(\alpha)/\partial\alpha$, $a = \hat{\beta}h(0)$, $b = 0$. Suppose A is empty. By applying [Lemma 1, P. 22] with $\kappa = \infty$, we have $h(\alpha) < \hat{\beta}h(0)$ for all $\alpha \geq 0$. Then for every $\tilde{\alpha} > 0$, we have

$$g(\tilde{\alpha}) = g(0) + \int_0^{\tilde{\alpha}} h(\alpha)d\alpha < g(0) + \hat{\beta}h(0)\tilde{\alpha},$$

which goes to $-\infty$ as $\tilde{\alpha} \to \infty$. Since g is bounded below, so A is not empty. Moreover, A is closed and bounded below. Therefore, we have $\hat{\alpha} \in A$.

Since $h(0) < \hat{\beta}h(0)$ and h is continuous, applying [Lemma 2, P. 20] with $a = \hat{\beta}h(0)$, and $\kappa = \infty$, there exists some $\epsilon > 0$ so that $h(\alpha) < \hat{\beta}h(0)$ for all $\alpha \in [0, \epsilon]$. As a result, we have $\hat{\alpha} > 0$. Moreover, we can show that $h(\hat{\alpha}) = \hat{\beta}h(0)$. Indeed, applying again [Lemma 1, P. 22] with $\kappa = \hat{\alpha}$, $a = \hat{\beta}h(0)$, and $b = 0$, we have $h(\alpha) < \hat{\beta}h(0)$ for all $\alpha \leq \hat{\alpha}$. Consider the sequence $\{\hat{\alpha} - 1/k\}$. By continuity, we have $h(\hat{\alpha}) = \lim_{k\to\infty} h(\hat{\alpha} - 1/k) \leq \hat{\beta}h(0)$. However, we also have $h(\hat{\alpha}) \geq \hat{\beta}h(0)$. Therefore, we have $h(\hat{\alpha}) = \hat{\beta}h(0)$. Consequently, we have $|h(\hat{\alpha})| = |\partial g(\hat{\alpha})/\partial\alpha| < \beta|h(0)| = \beta|\partial g(0)/\partial\alpha|$, as $\hat{\beta} < \beta$. The continuity of h ensures the existence of δ_1.

P. 24 It is implicitly required in the definition that for all $\nabla f(x_k) \neq 0$, we have $\nabla f(x_k)'d_k < 0$.

P. 25

Lemma (1, P. 25)**.** *Let $\{a_k\}$ be a monotonically nonincreasing real sequence that converges to $\bar{a} \in \Re$. Then* $\inf\{a_k\} = \bar{a}$.

Proof. If $\{a_k\}$ is unbounded below, there exists some $\bar{k}$ such that $a_k < \bar{a} - 1$ for all $k \geq \bar{k}$, which contradicts with $a_k \to \bar{a}$. Therefore, $\inf\{a_k\}$ is finite. From the definition of limit and monotonicity of $\{a_k\}$, we have $\inf\{a_k\} = \bar{a}$. Q.E.D.

Lemma (2, P. 25)**.** *Let $\{a_k\}$ be a monotone real sequence. Suppose that $\{a_k\}$ has a convergent subsequence with its limit denoted as $\bar{a} \in \Re$. Then $a_k \to \bar{a}$.*

Proof. Suppose that $\{a_k\}$ is monotonically nonincreasing. Denote by $\{a_k\}_K$ be its convergent subsequence. Then we have $a_k \geq \bar{a}$ for all $k \in K$ in view of [Lemma 1, P. 25]. Moreover, since $\{a_k\}$ is monotonically nonincreasing, for every k, there exists some $\tilde{k} > k$ and $\tilde{k} \in K$ such that $a_k \geq a_{\tilde{k}} \geq \bar{a}$. Therefore, $a_k \geq \bar{a}$ for all k. Since $\lim_{k\to\infty, k\in K} a_k = \bar{a}$, for every $\epsilon > 0$, there exists some $\bar{k}$ such that $|a_k - a| = a_k - a \leq a_{\bar{k}} - a \leq \epsilon$ for all $k \geq \bar{k}$ and $k \in K$. Then for all $k \geq \bar{k}$, we have $|a_k - a| = a_k - a \leq a_{\bar{k}} - a \leq \epsilon$. Q.E.D.

Applying [Lemma 2, P. 25] with $a_k = f(x_k)$, we have $f(x_k)$ converges to $f(\bar{x})$.

P. 25 Since $\{\alpha_k\}_K$ converges to 0, there exists some $\bar{k}$ such that $\alpha_k \leq \beta s$ for all $k \geq \bar{k}$ and $k \in K$. Therefore, $\alpha_k/\beta \in \{s, \beta s, \beta^2 s, \dots\}$ for all $k \geq \bar{k}$ and $k \in K$. By the definition of Armijo rule, we have

$$f(x_k) - f[x_k + (\alpha_k/\beta)d_k] < -\sigma(\alpha_k/\beta)\nabla f(x_k)' d_k \quad \text{for all } k \geq \bar{k} \text{ and } k \in K.$$

P. 25 By mean value theorem, for every $k \in \overline{K}$, there exists some $\gamma_k \in (0,1)$ such that $f(x_k + \bar{\alpha}_k p_k) = f(x_k) + \nabla f(x_k + \gamma_k \bar{\alpha}_k p_k)' \bar{\alpha}_k p_k$. As a result,

$$\frac{f(x_k) - f(x_k + \bar{\alpha}_k p_k)}{\bar{\alpha}_k} = \frac{-\nabla f(x_k + \gamma_k \bar{\alpha}_k p_k)' \bar{\alpha}_k p_k}{\bar{\alpha}_k} = -\nabla f(x_k + \gamma_k \bar{\alpha}_k p_k)' p_k.$$

Since $\gamma_k \in (0,1)$ and $\lim_{k\to\infty, k\in\overline{K}} \bar{\alpha}_k = 0$, the limit of $-\nabla f(x_k + \gamma_k \bar{\alpha}_k p_k)' p_k$ is $-\nabla f(\bar{x})' \bar{p}$.

P. 26

Lemma (1, P. 26)**.** *Suppose that $\{a_k\}$ and $\{b_k\}$ are two convergent real sequences with their limits denoted by a and b. Then we have $\{a_k b_k\}$ converges to ab.*

Proof. We have $|a_k b_k - ab| = |(a_k b_k - a b_k) + (a b_k - ab)| \leq |b_k||a_k - a| + |a||b_k - b|$. Since $|b_k| \leq c$ for some $c > 0$, for every ϵ, there exists some $\bar{k}$ such that $c|a_k - a| + |a||b_k - b| \leq \epsilon$. Q.E.D.

For every k, we have

$$\frac{-\nabla f(x_k)' d_k}{|d_k|} = \frac{|\nabla f(x_k)' d_k|}{|d_k|} \geq \frac{\inf_{i\geq k} |\nabla f(x_i)' d_i|}{\sup_{j \geq k} |d_j|}.$$

Applying [Lemma 1, P. 26] with $a_k = \inf_{i\geq k} |\nabla f(x_i)' d_i|$ and $b_k = 1/\sup_{j\geq k} |d_j|$, we obtain the desired inequality.

P. 27 By first-order Taylor series expansion, we have

$$f(x_k + \alpha d_k) = f(x_k) + \int_0^1 \nabla f(x_k + \tau \alpha d_k)' \alpha d_k d\tau.$$

Applying change of variable in integration with $\tau = t/\alpha$, there holds

$$\int_0^{\alpha} \nabla f(x_k + td_k)'\alpha d_k(1/\alpha)dt = \int_0^{\alpha} \nabla f(x_k + td_k)'d_k dt.$$

P. 29 Let us denote by $g_i(x)$ the ith element of $\nabla f(x)$, i.e., $g_i(x) = \partial f(x)/\partial x_i$. As a result, $\nabla g_i(x)'$ is the ith row of the matrix $\nabla^2 f(x)$. Moreover, $g_i(x^*) = 0$ for all i. Suppose that $x \in S(x^*;\delta)$. Applying first-order Taylor series expansion of g_i around x^*, we have[2]

$$\begin{aligned} g_i(x) =& g_i(x^*) + \int_0^1 \nabla g_i[x^* + \alpha(x - x^*)]'(x - x^*)d\alpha \\ =& \int_0^1 \nabla g_i[x^* + \alpha(x - x^*)]'(x - x^*)d\alpha. \end{aligned}$$

Then we multiply on both sides $(x_i - x_i^*)$, where x_i and x_i^* denote the ith elements of x and x^*, respectively, and sum over i. For the left hand side, we obtain $\sum_{i=1}^n (x_i - x_i^*)g_i(x) = \nabla f(x)'(x - x^*)$. For the right hand side, we have

$$\begin{aligned} &\sum_{i=1}^n (x_i - x_i^*) \int_0^1 \nabla g_i[x^* + \alpha(x - x^*)]'(x - x^*)d\alpha \\ =& \sum_{i=1}^n \int_0^1 (x_i - x_i^*)\nabla g_i[x^* + \alpha(x - x^*)]'(x - x^*)d\alpha \\ =& \int_0^1 \sum_{i=1}^n (x_i - x_i^*)\nabla g_i[x^* + \alpha(x - x^*)]'(x - x^*)d\alpha \\ =& \int_0^1 (x - x^*)'\nabla^2 f[x^* + \alpha(x - x^*)](x - x^*)d\alpha \\ \geq& m|x - x^*|^2 \end{aligned}$$

where the last inequality is due to that $x^* + \alpha(x - x^*) \in S(x^*;\delta)$ for all $\alpha \in (0,1)$. Therefore, we have $\nabla f(x)'(x - x^*) \geq m|x - x^*|^2$. If $|\nabla f(x)| < \epsilon$, then $\epsilon|x - x^*| \geq m|x - x^*|^2$, which implies that $|x - x^*| \leq \epsilon/m$.

Next, applying mean value theorem for $f \in C^2$ around x^*, for some $\alpha \in (0,1)$, we have

$$f(x) - f(x^*) = \frac{1}{2}(x - x^*)\nabla^2 f[x^* + \alpha(x - x^*)](x - x^*) \leq \frac{M}{2}|x - x^*|^2 \leq \frac{M\epsilon^2}{2m}.$$

[2]By stacking over i the values $g_i(x)$ and $g_i(x^*) + \int_0^1 \nabla g_i[x^* + \alpha(x - x^*)]'(x - x^*)d\alpha$ to form corresponding vectors, we obtain first-order Taylor expansion for vector-valued function given as

$$\nabla f(x) = \nabla f(x^*) + \int_0^1 \nabla^2 f[x^* + \alpha(x - x^*)](x - x^*)d\alpha.$$

Note that although x^* is a local minimum, it is not given that $f(x^*) \leq f(\hat{x})$ for all $\hat{x} \in S(x^*; \delta)$. Our derivation only relies on the condition $\nabla f(x^*) = 0$.

P. 30

Lemma (1, P. 30)**.** *Let $g : X \times Z \mapsto \Re$ be a continuous function, where X is a subset of $\Re^n$, and Z is a compact subset of $\Re^m$. Let $h : X \mapsto \Re$ be defined as*

$$h(x) = \min_{z \in Z} g(x, z).$$

Then we have that h is continuous over X.

Proof. Suppose that $\{x_k\} \subset X$ is a convergent sequence such that $x_k \to \bar{x} \in X$. For all k, there exists some $z_k \in Z$ such that $g(x_k, z_k) = h(x_k)$. Since Z is a compact set, there exists a subsequence $\{z_k\}_K$ that is convergent, with its limit denoted by $\bar{z}$. By the definition of h, we have

$$h(\bar{x}) \leq g(\bar{x}, \bar{z}). \tag{2}$$

Moreover, for every $\epsilon > 0$, there exists some $\bar{k}$ such that $|g(x_k, z_k) - g(\bar{x}, \bar{z})| < \epsilon$ for all $k \geq \bar{k}$ and $k \in K$. This inequality implies that

$$g(\bar{x}, \bar{z}) < g(x_k, z_k) + \epsilon \quad \text{for all } k \geq \bar{k} \text{ and } k \in K. \tag{3}$$

Combining Eqs. (2) and (3), we have

$$h(\bar{x}) < g(x_k, z_k) + \epsilon \quad \text{for all } k \geq \bar{k} \text{ and } k \in K. \tag{4}$$

Let us assume that the function h is discontinuous at $\bar{x}$. Then there exists some $\epsilon > 0$, such that for all $\hat{k} \in K$, there exists some $k \geq \hat{k}$ and $k \in K$ such that $|h(x_k) - h(\bar{x})| \geq \epsilon$. In other words, there exists a subsequence $\{x_k\}_{\overline{K}}$ of $\{x_k\}_K$ such that $|h(\bar{x}) - h(x_k)| = |h(\bar{x}) - g(x_k, z_k)| \geq \epsilon$ for all $k \in \overline{K}$. Equivalently, for every $k \in \overline{K}$, we have

$$h(\bar{x}) - g(x_k, z_k) \geq \epsilon \quad \text{or} \quad g(x_k, z_k) - h(\bar{x}) \geq \epsilon. \tag{5}$$

Combining Eqs. (4) and (5), we have

$$h(\bar{x}) \leq g(x_k, z_k) - \epsilon \quad \text{for all } k \geq \bar{k} \text{ and } k \in \overline{K}. \tag{6}$$

Let us denote by $\hat{z}$ the minimizer at $\bar{x}$, i.e., $g(\bar{x}, \hat{z}) = h(\bar{x})$. By the continuity of g, there exists some $\tilde{k} \geq \bar{k}$ and $\tilde{k} \in \overline{K}$ such that

$$g(x_k, \hat{z}) \leq h(\bar{x}) + \frac{\epsilon}{2} \quad \text{for all } k \geq \tilde{k} \text{ and } k \in \overline{K}.$$

Together with Eq. (6), this implies that $g(x_k, \hat{z}) < h(x_k)$, which is a contradiction. Q.E.D.

Let us define the function $g(x,z) = z'\nabla^2 f(x)z$, with $X = \Re^n$ and $Z = \{z \mid |z| = 1\}$. By [Lemma 1, P. 30], we have that $h(x) = \min_{z\in Z} g(x,z)$ is continuous. Since $h(x^*) > 0$, there exists some $\bar{\epsilon}$ such that $h(x) > 0$ for all $x \in S(x^*;\bar{\epsilon})$.

P. 30

Lemma (2, P. 30)**.** *Let $h : \Re \mapsto \Re$ be a continuous function. We have*

$$\left(\int_0^1 h(t)dt\right)^2 \le \int_0^1 \big(h(t)\big)^2 dt.$$

Proof. There holds

$$0 \le \int_0^1 \big(h(t) - c\big)^2 dt = \int_0^1 \big(h(t)\big)^2 dt - 2c\int_0^1 h(t)dt + c^2.$$

By setting $c = \int_0^1 h(t)dt$, we obtain the desired inequality. Q.E.D.

Lemma (3, P. 30)**.** *Let $g : \Re \mapsto \Re^n$ be a continuous function. We have*

$$\left(\int_0^1 g(t)dt\right)'\left(\int_0^1 g(t)dt\right) \le \int_0^1 g(t)'g(t)dt.$$

Proof. Let us denote by $g_i(t)$ the ith component of $g(t)$. Then we have

$$\left(\int_0^1 g(t)dt\right)'\left(\int_0^1 g(t)dt\right) = \sum_{i=1}^n \left(\int_0^1 g_i(t)dt\right)^2,$$

and

$$\int_0^1 g(t)'g(t)dt = \int_0^1 \sum_{i=1}^n \big(g_i(t)\big)^2 dt = \sum_{i=1}^n \int_0^1 \big(g_i(t)\big)^2 dt.$$

Applying [Lemma 2, P. 30] for each g_i and summing over i, we obtain the desired inequality. Q.E.D.

Applying first-order Taylor expansion to vector-valued function (see [P. 29]), we have

$$\nabla f(x_{\bar{k}}) = \int_0^1 \nabla^2 f[x^* + t(x_{\bar{k}} - x^*)](x_{\bar{k}} - x^*)dt.$$

Clearly, we have

$$|\nabla f(x_{\bar{k}})|^2 = \left(\int_0^1 g(t)dt\right)'\left(\int_0^1 g(t)dt\right),$$

where

$$g(t) = \nabla^2 f[x^* + t(x_{\bar{k}} - x^*)](x_{\bar{k}} - x^*).$$

Applying [Lemma 3, P. 30], we have

$$\begin{aligned}|\nabla f(x_{\bar{k}})|^2 \leq& \int_0^1 g(t)'g(t)dt\\=&\int_0^1 (x_{\bar{k}} - x^*)'\big(\nabla^2 f[x^* + t(x_{\bar{k}} - x^*)]\big)^2(x_{\bar{k}} - x^*)dt\\\leq&\int_0^1 \Gamma^2|x_{\bar{k}} - x^*|^2 dt\\=&\Gamma^2|x_{\bar{k}} - x^*|^2.\end{aligned}$$

A slight generalization is given as [Lemma 1, P. 83].

P. 30 Suppose there exists some $\tilde{x} \in S(x^*;\bar{\epsilon})$ such that $\tilde{x} \neq x^*$ and $\nabla f(\tilde{x}) = 0$. By mean value theorem, we have

$$f(x^*) = f(\tilde{x}) + \frac{1}{2}(x^* - \tilde{x})'\nabla^2 f[\tilde{x} + \alpha(x^* - \tilde{x})](x^* - \tilde{x})$$

for some $\alpha \in (0,1)$, which implies that $f(x^*) > f(\tilde{x})$. On the other hand, we also have

$$f(\tilde{x}) = f(x^*) + \frac{1}{2}(\tilde{x} - x^*)'\nabla^2 f[x^* + \beta(\tilde{x} - x^*)](\tilde{x} - x^*)$$

for some $\beta \in (0,1)$ so that $f(\tilde{x}) > f(x^*)$. This is a contradiction. As a result, x^* is the unique critical point within $S(x^*;\bar{\epsilon})$.

Suppose now that x_k does not converge to x^*. Then there exists a subsequence $\{x_k\}_K$ such that $|x_k - x^*| > \epsilon$ for all $k \in K$ and some $\epsilon > 0$. Since $\overline{L}$ is compact, there exists a subsequence $\{x_k\}_{\overline{K}}$ of $\{x_k\}_K$ that is convergent. By assumption, its limit, denoted by $\tilde{x}$, belongs to $\overline{L} \subset S(x^*;\bar{\epsilon})$,[3] and is also a critical point. Besides, $\tilde{x}$ is not x^* since $|x_k - x^*| > \epsilon$ for all $k \in \overline{K}$. This is a contradiction as there is a unique critical point within $S(x^*;\bar{\epsilon})$.

P. 31

Lemma (1, P. 31)**.** *Let A_k be a sequence of n by n symmetric matrices that converges to* 0*. Consider the sequences $\{y_k\}$ and $\{z_k\}$ defined as $y_k = \min_{|w|=1} w'A_k w$ and $z_k = \max_{|w|=1} w'A_k w$, respectively. There holds that $y_k \to 0$ and $z_k \to 0$.*

Proof. Denote by a_{ij}^k the ijth element of A_k and by w_i the ith element of a vector w. Then we have

$$|w'A_k w| = \left|\sum_{i=1}^n\sum_{j=1}^n a_{ij}^k w_i w_j\right| \leq \sum_{i=1}^n\sum_{j=1}^n |a_{ij}^k|$$

[3]To see this, we note that the for all $x \in L$, we have $|x - x^*| < \bar{\epsilon}/(1 + cs\Gamma) < \bar{\epsilon}$, where the last inequality is due to $c > 0$, $s > 0$, and $\Gamma > 0$. As a result, if $x \in \overline{L}$, we have $|x - x^*| \leq \bar{\epsilon}/(1 + cs\Gamma) < \bar{\epsilon}$, which means $\overline{L} \subset S(x^*;\bar{\epsilon})$.

for all w with $|w| = 1$. Therefore, we have

$$-\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}^k| \leq y_k \leq \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}^k|.$$

Since A_k converges to 0, we have $y_k \to 0$. The convergence of $\{z_k\}$ can be proven similarly. Q.E.D.

A formal definition of $o(\cdot)$ can be found in [NLP 3rd, P. 752]. Let us denote by $h(x, x^*)$ the function $o(|x - x^*|)$. By mean-value theorem, we have

$$h(x, x^*) = \frac{1}{2}(x - x^*)'\Big(\nabla^2 f[x^* + \alpha(x, x^*)(x - x^*)] - \nabla^2 f(x^*)\Big)(x - x^*),$$

where $\alpha(x, x^*) \in [0, 1]$. Let $\{x_k\}$ be a sequence such that $x_k \to x^*$. We have

$$h(x_k, x^*) = \frac{|x_k - x^*|^2}{2} w_k' A_k w_k,$$

where $A_k = \nabla^2 f[x^* + \alpha(x_k, x^*)(x_k - x^*)] - \nabla^2 f(x^*)$ and $w_k = (x_k - x^*)/|x_k - x^*|$. Since $f \in C^2$, $\alpha(x_k, x^*) \in [0, 1]$, $x_k \to x^*$, and

$$|x^* + \alpha(x_k, x^*)(x_k - x^*) - x^*| = \alpha(x_k, x^*)|x_k - x^*| \leq |x_k - x^*|,$$

we have A_k converges to 0. Applying [Lemma 1, P. 31], we have $w_k' A_k w_k$ converges to 0, which means that $h(x_k, x^*)/|x_k - x^*|^2$ converges to 0.

P. 32

Lemma (1, P. 32)**.** *Let λ_i, $i = 1, 2, \ldots, n$, be some positive real numbers so that $0 < m \leq \lambda_i \leq M$, $i = 1, 2, \ldots, n$, for some m and M. In addition, suppose that some nonnegative real numbers a_i, $i = 1, 2, \ldots, n$, satisfy that $\sum_{i=1}^{n} a_i \leq 1$. There holds*

$$\left(\sum_{i=1}^{n} a_i\lambda_i\right)\left(\sum_{i=1}^{n}\frac{a_i}{\lambda_i}\right) \leq \frac{(M+m)^2}{4Mm}.$$

Proof. Since $0 < m \leq \lambda_i \leq M$ for all i, we have $(\lambda_i - m)(M - \lambda_i) \geq 0$, which is $\lambda_i^2 + Mm \leq \lambda_i(M + m)$. Dividing on both sides λ_i, we have $\lambda_i + (Mm)/\lambda_i \leq M + m$. Multiplying on both sides a_i and adding over i, we have

$$\sum_{i=1}^{n} a_i\lambda_i + Mm\sum_{i=1}^{n}\frac{a_i}{\lambda_i} \leq M + m, \tag{7}$$

where the last inequality is due to that $M + m > 0$ and $\sum_{i=1}^{n} a_i \leq 1$.

Since for all real numbers u and v, we have $4uv \leq (u + v)^2$, by setting $u = \sum_{i=1}^{n} a_i\lambda_i$ and $v = Mm\sum_{i=1}^{n} a_i/\lambda_i$, we have

$$4\left(\sum_{i=1}^{n} a_i\lambda_i\right)\left(Mm\sum_{i=1}^{n}\frac{a_i}{\lambda_i}\right) \leq \left(\sum_{i=1}^{n} a_i\lambda_i + Mm\sum_{i=1}^{n}\frac{a_i}{\lambda_i}\right)^2. \tag{8}$$

Combining Eqs. (7) and (8), we get the desired inequality. Q.E.D.

The following result is known as Kantorovich Inequality.

Lemma (2, P. 32). *Let L be a positive definite symmetric $n \times n$ matrix. Then for any vector $y \in \Re^n$, $y \neq 0$, there holds*

$$\frac{(y'y)^2}{(y'Ly)(y'L^{-1}y)} \geq \frac{4Mm}{(M+m)^2}.$$

Proof. Since L is symmetric and positive definite, there exists some orthogonal matrix Q such that $L = Q'DQ$, where D is a diagonal matrix, with its diagonal elements λ_i, $i = 1, \ldots, n$, being the eigenvalues of L. Then we also have $L^{-1} = Q'D^{-1}D$. Denote $x = Qy$. Since Q is orthogonal, we have $y'y = y'Q'Qy = x'x$. Moreover,

$$y'Ly = x'Dx = \sum_{i=1}^n x_i^2 \lambda_i, \quad y'L^{-1}y = x'D^{-1}x = \sum_{i=1}^n \frac{x_i^2}{\lambda_i}.$$

where x_i denotes the ith component of x. As a result, we have

$$\frac{(y'Ly)(y'L^{-1}y)}{(y'y)^2} = \frac{\left(\sum_{i=1}^n x_i^2 \lambda_i\right)\left(\sum_{i=1}^n \frac{x_i^2}{\lambda_i}\right)}{\left(\sum_{i=1}^n x_i\right)^2}.$$

By setting $a_i = x_i^2 / \left(\sum_{i=1}^n x_i\right)$, we obtain

$$\frac{(y'Ly)(y'L^{-1}y)}{(y'y)^2} = \left(\sum_{i=1}^n a_i \lambda_i\right)\left(\sum_{i=1}^n \frac{a_i}{\lambda_i}\right).$$

Applying [Lemma 1, P. 32], we obtain

$$\frac{(y'Ly)(y'L^{-1}y)}{(y'y)^2} \leq \frac{(M+m)^2}{4Mm},$$

which is equivalent to the desired inequality. Q.E.D.

P. 36 To see that $D_k \nabla f(x_k) \to 0$, we note that

$$D_k \nabla f(x_k) = \frac{|\nabla f(x_k)| \left(D_k \nabla f(x_k) - \left[\nabla^2 f(x^*)\right]^{-1} \nabla f(x_k)\right)}{|\nabla f(x_k)|} + \left[\nabla^2 f(x^*)\right]^{-1} \nabla f(x_k).$$

As a result,

$$|D_k \nabla f(x_k)| \leq \frac{|\nabla f(x_k)| \left|D_k \nabla f(x_k) - \left[\nabla^2 f(x^*)\right]^{-1} \nabla f(x_k)\right|}{|\nabla f(x_k)|} + \left|\left[\nabla^2 f(x^*)\right]^{-1}\right| |\nabla f(x_k)|.$$

In view of Eqs. (28) and (30) in [COLMM], we have $D_k \nabla f(x_k) \to 0$.

P. 36 Let us provide a detailed calculation for the value γ_k. Since $D_k p_k = \big[\nabla^2 f(x^*)\big]^{-1} p_k + \beta_k$, we have

$$\begin{aligned}(1-\sigma)p_k' D_k p_k =&(1-\sigma)p_k'\Big(\big[\nabla^2 f(x^*)\big]^{-1} p_k + \beta_k\Big)\\ =&(1-\sigma)p_k'\big[\nabla^2 f(x^*)\big]^{-1} p_k + (1-\sigma)p_k'\beta_k.\end{aligned}$$

Since β_k converges to zero and $\{p_k\}$ is bounded, we have $(1-\sigma)p_k'\beta_k$ converges to 0. On the other hand, we have

$$\begin{aligned}&\frac{1}{2}p_k' D_k \nabla^2 f(\bar{x}_k) D_k p_k\\ =&\frac{1}{2}\Big(\big[\nabla^2 f(x^*)\big]^{-1} p_k + \beta_k\Big)'\nabla^2 f(\bar{x}_k)\Big(\big[\nabla^2 f(x^*)\big]^{-1} p_k + \beta_k\Big)\\ =&\frac{1}{2}p_k'\big[\nabla^2 f(x^*)\big]^{-1}\nabla^2 f(\bar{x}_k)\big[\nabla^2 f(x^*)\big]^{-1} p_k +\\ &\beta_k'\nabla^2 f(\bar{x}_k)\big[\nabla^2 f(x^*)\big]^{-1} p_k + \frac{1}{2}\beta_k'\nabla^2 f(\bar{x}_k)\beta_k\\ =&\frac{1}{2}p_k'\big[\nabla^2 f(x^*)\big]^{-1}\nabla^2 f(x^*)\big[\nabla^2 f(x^*)\big]^{-1} p_k +\\ &\frac{1}{2}p_k'\big[\nabla^2 f(x^*)\big]^{-1}\big(\nabla^2 f(\bar{x}_k) - \nabla^2 f(x^*)\big)\big[\nabla^2 f(x^*)\big]^{-1} p_k +\\ &\beta_k'\nabla^2 f(\bar{x}_k)\big[\nabla^2 f(x^*)\big]^{-1} p_k + \frac{1}{2}\beta_k'\nabla^2 f(\bar{x}_k)\beta_k.\end{aligned}$$

Applying [Lemma 1, P. 31] with $A_k = \big(\nabla^2 f(\bar{x}_k) - \nabla^2 f(x^*)\big)$, we can show that

$$\frac{1}{2}p_k'\big[\nabla^2 f(x^*)\big]^{-1}\big(\nabla^2 f(\bar{x}_k) - \nabla^2 f(x^*)\big)\big[\nabla^2 f(x^*)\big]^{-1} p_k \to 0.$$

Moreover, $\beta_k'\nabla^2 f(\bar{x}_k)\big[\nabla^2 f(x^*)\big]^{-1} p_k \to 0$ and $\frac{1}{2}\beta_k'\nabla^2 f(\bar{x}_k)\beta_k \to 0$. As a result, we have $\gamma_k \to 0$, where

$$\begin{aligned}\gamma_k =&\frac{1}{2}p_k'\big[\nabla^2 f(x^*)\big]^{-1}\big(\nabla^2 f(\bar{x}_k) - \nabla^2 f(x^*)\big)\big[\nabla^2 f(x^*)\big]^{-1} p_k +\\ &\beta_k'\nabla^2 f(\bar{x}_k)\big[\nabla^2 f(x^*)\big]^{-1} p_k + \frac{1}{2}\beta_k'\nabla^2 f(\bar{x}_k)\beta_k - (1-\sigma)p_k'\beta_k.\end{aligned}$$

P. 37 We state and prove Taylor's theorem for vector-valued functions, using mean value theorem for scalar-valued functions.

Lemma (1, P. 37)**.** *Let $g : \Re^n \mapsto \Re^m$ be a continuously differentiable function, with its component functions denoted as g_i, $i = 1, \ldots, m$. For all $x, \tilde{x} \in \Re^n$, define $h(x, \tilde{x}) = g(x) - g(\tilde{x}) - \big[\nabla g(\tilde{x})\big]'(x - \tilde{x})$. For every sequence $\{x_k\}$ such that $x_k \to \tilde{x}$, we have $h(x_k, \tilde{x})/|x_k - \tilde{x}|$ converges to 0.*

Proof. Applying mean value theorem to ith component function g_i, we have

$$\begin{aligned}g_i(x) =&g_i(\tilde{x}) + \nabla g_i\big[\tilde{x} + \alpha_i(x,\tilde{x})(x - \tilde{x})\big]'(x - \tilde{x})\\ =&g_i(\tilde{x}) + \nabla g_i(\tilde{x})'(x - \tilde{x}) + \Big(\nabla g_i\big[\tilde{x} + \alpha_i(x,\tilde{x})(x - \tilde{x})\big] - \nabla g_i(\tilde{x})\Big)'(x - \tilde{x}),\end{aligned}$$

where $\alpha(x,\tilde{x}) \in [0,1]$. Clearly, we have

$$h_i(x,\tilde{x}) = \Big(\nabla g_i\big[\tilde{x} + \alpha_i(x,\tilde{x})(x-\tilde{x})\big] - \nabla g_i(\tilde{x})\Big)'(x-\tilde{x}), \quad i = 1,\dots,m,$$

where h_i is the component function of h. Let $\{x_k\}$ be a sequence such that $x_k \to \tilde{x}$. Then we have $\tilde{x} + \alpha_i(x_k,\tilde{x})(x_k - \tilde{x})$ converges to $\tilde{x}$ as $\alpha(x_k,\tilde{x}) \in [0,1]$. As a result, we have

$$\begin{aligned}
\frac{|h_i(x_k,\tilde{x})|}{|x_k-\tilde{x}|} &= \frac{\Big|\Big(\nabla g_i\big[\tilde{x}_k + \alpha_i(x_k,\tilde{x})(x_k-\tilde{x})\big] - \nabla g_i(\tilde{x})\Big)'(x_k-\tilde{x})\Big|}{|x_k-\tilde{x}|} \\
&\le \frac{\Big|\nabla g_i\big[\tilde{x}_k + \alpha_i(x_k,\tilde{x})(x_k-\tilde{x})\big] - \nabla g_i(\tilde{x})\Big| |(x_k-\tilde{x})|}{|x_k-\tilde{x}|} \\
&= \Big|\nabla g_i\big[\tilde{x}_k + \alpha_i(x_k,\tilde{x})(x_k-\tilde{x})\big] - \nabla g_i(\tilde{x})\Big|,
\end{aligned}$$

which converges to 0 as $k \to \infty$. Q.E.D.

Note that we have $\delta_k = \beta_k$ for all k. As a result,

$$D_k \nabla f(x_k) = \big[\nabla^2 f(x^*)\big]^{-1} \nabla f(x_k) + |\nabla f(x_k)|\beta_k. \tag{9}$$

By using [Lemma 1, P. 37] for the function ∇f, we have $\nabla f(x_k) = \nabla^2 f(x^*)(x_k - x^*) + h(x_k,x^*)$, where $h(x_k,x^*)/|x_k - x^*|$ converges to 0 as $k \to \infty$. Then we have

$$\begin{aligned}
\big[\nabla^2 f(x^*)\big]^{-1} \nabla f(x_k) &= x_k - x^* + \big[\nabla^2 f(x^*)\big]^{-1} h(x_k,x^*), \\
|\nabla f(x_k)|\beta_k &= \big|\nabla^2 f(x^*)(x_k - x^*) + h(x_k,x^*)\big|\beta_k.
\end{aligned}$$

Using above two relations in Eq. (9), we obtain

$$D_k \nabla f(x_k) = x_k - x^* + \big[\nabla^2 f(x^*)\big]^{-1} h(x_k,x^*) + \big|\nabla^2 f(x^*)(x_k - x^*) + h(x_k,x^*)\big|\beta_k.$$

Since $x_{k+1} - x^* = x_k - x^* - D_k \nabla f(x_k)$, we obtain

$$x_{k+1} - x^* = -\big[\nabla^2 f(x^*)\big]^{-1} h(x_k,x^*) - \big|\nabla^2 f(x^*)(x_k - x^*) + h(x_k,x^*)\big|\beta_k.$$

As a result,

$$|x_{k+1} - x^*| \le \Big|\big[\nabla^2 f(x^*)\big]^{-1}\Big| |h(x_k,x^*)| + \big|\nabla^2 f(x^*)\big| |x_k - x^*| |\beta_k| + |h(x_k,x^*)| |\beta_k|.$$

P. 37

Lemma (2, P. 37)**.** *Suppose that $\{D_k\}$ is a sequence of n by n positive definite symmetric matrices, $\{g_k\}$ is a sequence of n-dimensional vectors with $g_k \neq 0$ for all k, and H be a positive definite matrix. Then we have*

$$\lim_{k\to\infty} \frac{|(D_k - H^{-1})g_k|}{|g_k|} = 0 \quad \textit{if and only if} \quad \lim_{k\to\infty} \frac{|(D_k^{-1} - H)D_k g_k|}{|D_k g_k|} = 0.$$

Proof. We first prove the only if part. We first note the following equalities hold:

$$(D_k^{-1} - H)D_k g_k = g_k - HD_k g_k = H(H^{-1}g_k - D_k g_k) = H(H^{-1} - D_k)g_k. \quad (10)$$

Then we have

$$\begin{aligned} \frac{|(D_k^{-1} - H)D_k g_k|}{|D_k g_k|} &= \frac{|H(H^{-1} - D_k)g_k|}{|D_k g_k|} \leq |H| \frac{|(D_k - H^{-1})g_k|}{|D_k g_k|} \\ &= |H| \frac{|(D_k - H^{-1})g_k|}{|g_k|} \frac{|g_k|}{|D_k g_k|}, \end{aligned} \quad (11)$$

where the first equality is due to Eq. (10). Since $D_k g_k = H^{-1}g_k + (D_k - H^{-1})g_k$, we have

$$\begin{aligned} \frac{|D_k g_k|}{|g_k|} &= \frac{|H^{-1}g_k + (D_k - H^{-1})g_k|}{|g_k|} \geq \frac{|H^{-1}g_k|}{|g_k|} - \frac{|(D_k - H^{-1})g_k|}{|g_k|} \\ &\geq m_1 - \frac{|(D_k - H^{-1})g_k|}{|g_k|}, \end{aligned}$$

where m_1 is the smallest eigenvalue of H^{-1}, which is positive. Since $|(D_k - H^{-1})g_k|/|g_k| \to 0$, there exists some $\bar{k}$ such that $|(D_k - H^{-1})g_k|/|g_k| \leq m_1/2$ for all $k \geq \bar{k}$. As a result, $|D_k g_k|/|g_k| \geq m_1/2$ for all $k \geq \bar{k}$, or equivalently, $|g_k|/|D_k g_k| \leq 2/m_1$. Together with Eq. (11), we obtain

$$\frac{|(D_k^{-1} - H)D_k g_k|}{|D_k g_k|} \leq |H| \frac{|(D_k - H^{-1})g_k|}{|g_k|} \frac{2}{m_1} \quad \text{for all } k \geq \bar{k}.$$

Taking limit on both sides concludes this part of the proof.

As for the if part, we note that

$$(D_k - H^{-1})g_k = H^{-1}(HD_k g_k - g_k) = H^{-1}(H - D_k^{-1})D_k g_k.$$

Using this equality similarly as the preceding part, we have

$$\frac{|(D_k - H^{-1})g_k|}{|g_k|} \leq |H^{-1}| \frac{|(D_k^{-1} - H)D_k g_k|}{|D_k g_k|} \frac{|D_k g_k|}{|g_k|}.$$

Using the equality $g_k = HD_k g_k + (D_k^{-1} - H)D_k g_k$, we can show, similarly as preceding part, that for some $\bar{k}$, there holds $|g_k|/|D_k g_k| \leq 2/m_2$ for all $k \geq \bar{k}$, where m_2 is the smallest eigenvalue of H. We can then proceed and prove the if part similarly. Q.E.D.

P. 42

Lemma (1, P. 42)**.** *Let $\{A_k\}$ be a sequence of n by n invertible matrices, whose limit A is also invertible. Then the sequence $\{A_k^{-1}\}$ converges to A^{-1}.*

Proof. Denote by d_k the determinant of A_k and by B_k the cofactor matrix of A_k. Then we have $d_k \to d$ and $B_k \to B$, where d and B denote the determinant and the cofactor matrix of A, respectively. According to the formulas of determinant and cofactor matrix, we have $d_k \to d$ and $B_k \to B$. Since $A_k^{-1} = B_k'/d_k$ and $A^{-1} = B'/d$, we obtain that $\{A_k^{-1}\}$ converges to A^{-1}. Q.E.D.

Define as d the function such that $d(x)$ is the determinant of matrix $\nabla g(x)$. Since $g \in C^1$ and $\nabla g(x^*)$ is invertible, we have $d \in C^1$ and $d(x^*) \neq 0$. As a result, there exists some $\delta \in (0, \epsilon)$ such that $d(x) \neq 0$ for all $x \in S(x^*; \delta)$, which implies that $[\nabla g(x)]^{-1}$ exists for all $x \in S(x^*; \delta)$. [Lemma 1, P. 42] can be used to show that $[\nabla g(x)]^{-1}$ is continuous.

P. 42 Define $h(t) = \big\{\nabla g(x_k)' - \nabla g[x^* - t(x_k - x^*)]\big\}'(x_k - x^*)$. Applying [Lemma 3, P. 30] using h in place of g, we have $\big(\int_0^1 h(t)dt\big)'\big(\int_0^1 h(t)dt\big) \le \int_0^1 h(t)'h(t)dt$. Equivalently, we have

$$\left|\int_0^1 \big\{\nabla g(x_k)' - \nabla g[x^* + t(x_k - x^*)]\big\}'(x_k - x^*)dt\right|^2$$
$$\le \int_0^1 \big|\big\{\nabla g(x_k)' - \nabla g[x^* + t(x_k - x^*)]\big\}'(x_k - x^*)\big|^2 dt.$$

Then the inequality follows as $\big|\big\{\nabla g(x_k)' - \nabla g[x^* + t(x_k - x^*)]\big\}'(x_k - x^*)\big| \le \big|\nabla g(x_k) - \nabla g[x^* + t(x_k - x^*)]\big|\big|(x_k - x^*)\big|$.

P. 43

Lemma (1, P. 43)**.** *Let $f : \Re^n \mapsto \Re^m$ be a continuous function and suppose that $\{x_k\} \subset \Re^n$ is a convergent sequence with its limit denoted by x^*. Then the sequence $\{y_k\} \subset \Re$ converges to 0, where*

$$y_k = \max_{\tilde{x} \in \{x \,|\, |x - x^*| \le |x_k - x^*|\}} |f(\tilde{x}) - f(x^*)|.$$

Proof. Suppose otherwise. Then there exists a subsequence $\{y_k\}_K$ such that $y_k > \epsilon$ for some $\epsilon > 0$. From the definition of y_k, for every $k \in K$, there exists some $\tilde{x}_k \in \{x \,|\, |x - x^*| \le |x_k - x^*|\}$ such that $|f(\tilde{x}_k) - f(x^*)| \ge \epsilon$. This contradicts with the continuity of f as $\tilde{x}_k$ also converges to x^*, in view of that $|\tilde{x}_k - x^*| \le |x_k - x^*|$. Q.E.D.

Applying [Lemma 1, P. 43] with ∇g as f, and since

$$\big|\nabla g(x_k) - \nabla g[x^* + t(x_k - x^*)]\big|$$
$$\le \big|\nabla g(x_k) - \nabla g(x^*)\big| + \big|\nabla g[x^* + t(x_k - x^*)] - \nabla g(x^*)\big| \le 2y_k$$

for all $t \in [0, 1]$, we have the sequence z_k converges to 0, where $z_k = \int_0^1 \big|\nabla g(x_k) - \nabla g[x^* + t(x_k - x^*)]\big|dt \le 2y_k$.

P. 43 Let y be n^2-dimensional vectors, and denote by y_i the n dimensional vector formed by the $[(i-1)n+1]$th to (in)th elements of y. Consider the function

$$h(y) = \big[\nabla g_1(y_1)\ \ldots\ \nabla g_n(y_n)\big]'\big[\nabla g_1(y_1)\ \ldots\ \nabla g_n(y_n)\big].$$

Since $\nabla g(x^*)$ is invertible, then $h(y^*)$ is positive definite, where $y_i^* = x^*$ for all i. Applying [Lemma 1, P. 30] by considering $z'h(y)z$, we have that there exists some $\tilde{\delta} > 0$ such that $\min_{|z|=1} z'h(y)z > 0$ for all $y \in S(y^*; \tilde{\delta})$. Setting $\delta_1 = \tilde{\delta}/\sqrt{n}$, we have

$$\sum_{i=1}^{n} |\tilde{x}_i - x^*|^2 \leq \sum_{i=1}^{n} |x - x^*|^2 \leq \tilde{\delta}^2 \quad \text{for all } x \in S(x^*; \delta_1).$$

As a result, we have $\nabla g(\tilde{x})\nabla g(\tilde{x})' = h(\tilde{y})$ being positive definite, where $\tilde{y}_i = \tilde{x}_i$.

P. 47 See footnote in [COLMM P. 63] for the definition of $O(\cdot)$.

P. 51 We discuss some properties of the Gram-Schmidt procedure. Suppose that $\xi_0, \ldots, \xi_i \in \Re^n$, where $i \leq n-1$ are some vectors and Q is a positive definite symmetric matrix. The Gram-Schmidt procedure starts by setting

$$d_0 = \xi_0. \tag{12}$$

At jth iteration, where $j = 1, \ldots, i$, we compute d_j as

$$d_j = \xi_j + \sum_{k=0}^{j-1} c_{jk} d_k, \tag{13}$$

where

$$c_{jk} = -\xi_j' Q d_k / (d_k' Q d_k), \quad k = 0, \ldots, j-1. \tag{14}$$

Lemma (1, P. 51)**.** *Let $\xi_0, \ldots, \xi_i$ be some vectors in $\Re^n$. Suppose that $\xi_0, \ldots, \xi_{i-1}$ are linearly independent.*

(a) The Gram-Schmidt procedure (12)-(14) *are well-defined in the sense that the vectors $d_0, \ldots, d_{i-1}$ are nonzero.*

(b) The vectors $d_0, \ldots, d_{i-1}$ are linearly independent.

(c) A vector v satisfies $v = \sum_{j=0}^{i-1} \alpha_j \xi_j$ for some α_j, $j = 0, \ldots, i-1$, if and only if $v = \sum_{j=0}^{i-1} \bar{\alpha}_j d_j$ for some $\bar{\alpha}_j$, $j = 0, \ldots, i-1$.

(d) A vector v satisfies $v'\xi_j = 0$ for all $j = 0, \ldots, i-1$, if and only if $v'd_j = 0$ for all $j = 0, \ldots, i-1$.

Proof. (a) We show that $d_j \neq 0$ for $j = 0, 1 \ldots, i-1$ by induction. Clearly, $d_0 = \xi_0 \neq 0$. Suppose that $d_0, \ldots, d_{j-1}$ are nonzero. The formulas for c_{jk}, $k =$

$0, 1 \ldots, j-1$, are well-defined (since $d_k' Q d_k \neq 0$) so that d_j is well-defined. To see that d_j is nonzero, we note that the vector $\sum_{k=0}^{j-1} c_{jk} d_k$ is a linear combination of the vectors $\xi_0, \ldots, \xi_{j-1}$. Therefore, $d_j = 0$ would imply that $\xi_0, \ldots, \xi_j$ are linearly dependent.

(b) Since the Gram-Schmidt procedure (12)-(14) are well-defined, we can write $d_j = \sum_{k=0}^{j} b_{kj} \xi_k$ for some coefficients b_{kj}, $k = 0, 1, \ldots, j$, where $b_{jj} = 1$. Consider the i by i upper triangular matrix B defined as

$$B = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0(i-1)} \\ & b_{11} & \cdots & b_{1(i-1)} \\ & & \ddots & \vdots \\ & & & b_{(i-1)(i-1)} \end{bmatrix}, \tag{15}$$

which is invertible as its diagonal elements are unity. Suppose there exists some $\beta_0, \ldots, \beta_{i-1}$, not all zero, such that $\sum_{j=0}^{i-1} \beta_j d_j = 0$, then we must have $\sum_{j=0}^{i-1} \bar{\beta}_j \xi_j = 0$, where the vector $[\bar{\beta}_0 \ldots \bar{\beta}_{i-1}]$ is given by $[\bar{\beta}_0 \ldots \bar{\beta}_{i-1}]' = B^{-1}[\beta_0 \ldots \beta_{i-1}]'$, which is a nonzero vector. This can not be true as $\xi_0, \ldots, \xi_{i-1}$ are linearly independent.

(c) Using the matrix B defined in Eq. (15), we have $[\bar{\alpha}_0 \ldots \bar{\alpha}_{i-1}]' = B^{-1}[\alpha_0 \ldots \alpha_{i-1}]'$.

(d) If v satisfies $v'\xi_j = 0$ for all $j = 0, \ldots, i-1$, then $v'd_j = \sum_{k=0}^{j} b_{kj} v' \xi_k = 0$ for $j = 0, \ldots, i-1$. Conversely, applying part (c), we have that the vectors $\xi_0, \ldots, \xi_{i-1}$ can be written as linear combinations of $d_0, \ldots, d_{i-1}$ and the conclusion follows similarly. Q.E.D.

Lemma (2, P. 51)**.** *Consider the Gram-Schmidt procedure* (12)-(14)*. Let* $\xi_0, \ldots, \xi_i$ *be some vectors in* $\Re^n$*. Suppose that* $\xi_0, \ldots, \xi_{i-1}$ *are linearly independent, and there exists some* $\alpha_0, \ldots, \alpha_{i-1}$ *such that* $\xi_i = \sum_{j=0}^{i-1} \alpha_j \xi_j$*. Then we have* $d_i = 0$*.*

Proof. Using the matrix B defined in Eq. (15), we have $\xi_i = \sum_{j=0}^{i-1} \bar{\alpha}_j d_j$, where $[\bar{\alpha}_0 \ldots \bar{\alpha}_{i-1}]' = B^{-1}[\alpha_0 \ldots \alpha_{i-1}]'$. As a result, $d_i = \sum_{j=0}^{i-1} \bar{c}_{ij} d_j$, where $\bar{c}_{ij} = \bar{\alpha}_j + c_{ij}$. Then we have

$$d_i' Q d_i = d_i' Q \left(\sum_{j=0}^{i-1} \bar{c}_{ij} d_j \right) = \sum_{j=0}^{i-1} \bar{c}_{ij} d_i' Q d_j = 0,$$

where the last equality is due to the construction rule of d_i. Since Q is positive definite, we have that d_i is zero. Q.E.D.

Note that a special case of [Lemma 2, P. 51] is where $\xi_i = 0$, where we also have $d_i = 0$.

P. 52

Lemma (1, P. 52)**.** *Let $g_0, \dots, g_i$ be some nonzero vectors such that $g_0, \dots, g_{i-1}$ are linearly independent. Then $g_0, \dots, g_i$ are linearly dependent if and only if there exists some $\alpha_0, \dots, \alpha_{i-1}$, not all zero, such that $g_i = \sum_{j=0}^{i-1} \alpha_j g_j$.*

Proof. The if part is obvious. To see the only if part, assume that $g_0, \dots, g_i$ is linearly dependent. Then there exists some β_j, $j = 0, \dots, i$, not all zero, such that $\sum_{j=0}^{i} \beta_j g_j = 0$. We must have $\beta_i \neq 0$ as otherwise it implies that $g_0, \dots, g_i$ are linearly dependent. We obtain the coefficients by setting $\alpha_j = -\beta_j/\beta_i$, $j = 0, \dots, i-1$. Q.E.D.

Lemma (2, P. 52)**.** *Let $g_0, \dots, g_i$ be some nonzero vectors such that $g_0, \dots, g_{i-1}$ are linearly independent. If $g_i' g_j = 0$ for $j = 0, \dots, i-1$, then $g_0, \dots, g_i$ are linearly independent.*

Proof. Assume that $g_0, \dots, g_i$ are linearly dependent. By [Lemma 1, P. 52], we have $g_i = \sum_{j=0}^{i-1} \alpha_j g_j$ for some $\alpha_0, \dots, \alpha_{i-1}$, not all zero. However, this implies $g_i' g_i = \sum_{j=0}^{i-1} \alpha_j g_i' g_j = 0$, which contradicts with g_i being nonzero. Q.E.D.

Lemma (3, P. 52)**.** *Starting with $g_0 \neq 0$, let $g_0, \dots, g_i$, $i \leq n-1$, be the vectors computed by the conjugate gradient method such that they are all nonzero. Then these vectors are linearly independent and the vectors $d_0, \dots, d_i$ are nonzero.*

Proof. We prove the statement by applying the conditions $g_j \neq 0$ one at a time. By construction, we have $d_0 = -g_0 \neq 0$, and $g_1' d_0 = -g_1' g_0 = 0$. Since $g_1 \neq 0$, this implies that g_0, g_1 are linearly independent. According to [Lemma 1(a), P. 51], this means that d_1 is nonzero.

Next, we apply the condition $g_2 \neq 0$. By [COLMM Prop. 1.18, P. 50], we have $g_2' d_j = 0$ for $j = 0, 1$. According to [Lemma 1(d), P. 51], this implies that $g_2' g_j = 0$ for $j = 0, 1$. Since we have shown that g_0, g_1 are linearly independent, applying [Lemma 2, P. 52], we obtain that g_0, g_1, g_2 are linearly independent and d_2 is nonzero.

We can proceed similarly to $g_0, \dots, g_i$, $i \leq n-1$. Q.E.D.

Lemma (4, P. 52)**.** *Starting with $g_0 \neq 0$, let $g_0, \dots, g_i$, $i \leq n-1$, be the vectors computed by the conjugate gradient method such that they are all nonzero. Then corresponding stepsize parameters $\alpha_0, \dots, \alpha_i$ are positive.*

Proof. According to [Lemma 3, P. 52], the vectors $d_0, \dots, d_i$ are nonzero. By construction of the conjugate gradient method, we have

$$(x_k + \alpha_k d_k)' Q d_k = 0, \qquad k = 0, \dots, i.$$

Since $d_k'Qd_k > 0$, the stepsize α_k being positive is equivalent to $x_k'Qd_k < 0$, $k = 0, \ldots, i$. Indeed, for every k,

$$x_k'Qd_k = g_k'd_k = g_k'(-g_k + \beta_k d_{k-1}) = -|g_k|^2,$$

where the formula for d_k is given by [COLMM Eq. (63), P. 52]. Since $g_k \neq 0$, we conclude the proof. Q.E.D.

From above discussion, it can be seen that if $g_k \neq 0$, we have

$$\alpha_k = \frac{|g_k|^2}{d_k'Qd_k}.$$

P. 53 This can be verified by checking the conditions stated in [COLMM Prop. 1.18, P. 50].

P. 54 In what follows, we show that $Qx_0, \ldots, Q^{k+1}x_0$ span the same subspace as $d_0, \ldots, d_k$.

Lemma (1, P. 54)**.** *Let $v_0, \ldots, v_i \in \Re^n$ be some vectors, where $i \leq n-1$, and $g_0, \ldots, g_i \in \Re^n$ be some vectors such that $g_j = \sum_{k=0}^{i} a_{kj} v_k$, for some coefficients a_{kj}, $j = 0, \ldots, i$. If the vectors $g_0, \ldots, g_i$ are linearly independent, then the matrix A is invertible, where*

$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0i} \\ a_{10} & a_{11} & \cdots & a_{1i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i0} & a_{i1} & \cdots & a_{ii} \end{bmatrix}.$$

Moreover, the vectors $v_0, \ldots, v_i$ are also linearly independent and span the same subspace as the vectors $g_0, \ldots, g_i$.

Proof. Suppose that A is not invertible. Then there exists some $\alpha_0, \ldots, \alpha_i$, not all zero, such that $A[\alpha_0 \ \ldots \ \alpha_i]' = 0$. As a result, we have $\sum_{j=0}^{i} \alpha_j g_j = [v_0 \ \ldots \ v_i]A[\alpha_0 \ \ldots \ \alpha_i]' = 0$, which contradicts with the fact that $g_0, \ldots, g_i$ are linearly independent. Therefore, A is invertible.

Suppose that there exists some $\beta_0, \ldots, \beta_i$, not all zero, such that $\sum_{j=0}^{i} \beta_j v_j = 0$. Then we have $\sum_{j=0}^{i} \bar{\beta}_j g_j = 0$, where $[\bar{\beta}_0 \ \ldots \ \bar{\beta}_i]' = A^{-1}[\beta_0 \ \ldots \ \beta_i]' \neq 0$. This contradicts with the fact that $g_0, \ldots, g_i$ are linearly independent. The remaining part of the proof can be established by using the fact that A is invertible. Q.E.D.

Lemma (2, P. 54)**.** *Starting with $g_0 = Qx_0 \neq 0$, let $g_0, \ldots, g_i$, be the vectors computed by the conjugate gradient method such that they are all nonzero. Let $d_0, \ldots, d_i$ be the corresponding Q-conjugate vectors.*

(a) For $j = 0, \dots, i$, we have

$$g_j = \sum_{k=0}^{j} a_{kj} Q^{k+1} x_0, \; d_j = \sum_{k=0}^{j} b_{kj} Q^{k+1} x_0, \quad \textit{for } k = 0, \dots, j \tag{16}$$

where a_{kj} and b_{kj}, $k = 0, \dots, j$, are some coefficients.

(b) The upper-triangular matrices A and B are invertible, where

$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0i} \\ & a_{11} & \cdots & a_{1i} \\ & & \ddots & \vdots \\ & & & a_{ii} \end{bmatrix}, \; B = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0i} \\ & b_{11} & \cdots & b_{1i} \\ & & \ddots & \vdots \\ & & & b_{ii} \end{bmatrix}.$$

(c) The vectors $Qx_0, \dots, Q^{i+1}x_0$ are linearly independent. Moreover, they span the same subspace as the vectors $d_0, \dots, d_i$.

Proof. (a) We show that Eq. (16) holds by induction. Clearly, for $j = 0$, we have $g_0 = Qx_0$ and $d_0 = -Qx_0$. Suppose next that Eq. (16) holds for $j = \ell$. Then for $j = \ell + 1$, we have

$$g_{\ell+1} = Q(x_\ell + \alpha_\ell d_\ell) = Qx_\ell + \alpha_\ell Q d_\ell = g_\ell + \alpha_\ell Q d_\ell.$$

Then the induction hypothesis yields

$$g_{\ell+1} = a_{0\ell} Q x_0 + \alpha_\ell b_{\ell\ell} Q^{\ell+2} x_0 + \sum_{k=1}^{\ell} (a_{k\ell} + \alpha_\ell b_{(k-1)\ell}) Q^{k+1} x_0 \tag{17}$$

Similarly, we can show it holds for $d_{\ell+1}$, as $d_{\ell+1} = -g_{\ell+1} + \beta_{\ell+1} d_\ell$ for some $\beta_{\ell+1}$; cf. [COLMM Eq. (63), P. 52].

(b) Since $g_0, \dots, g_i$ are nonzero, then by [Lemma 3, P. 52], we have that $g_0, \dots, g_i$ are linearly independent. Applying [Lemma 1, P. 54] by setting $v_j = Q^{j+1}x_0$, we have that A is invertible. Similarly, we have that B is invertible.

(c) As argued in part (b), we have that $g_0, \dots, g_i$ are linearly independent. The linear independence of $Qx_0, \dots, Q^{i+1}x_0$ follows by applying [Lemma 1, P. 54] with $v_j = Q^{j+1}x_0$. Moreover, we have that the vectors $Qx_0, \dots, Q^{i+1}x_0$ span the same subspace as $g_0, \dots, g_i$. Since $g_0, \dots, g_i$ and $d_0, \dots, d_i$ span the same subspace, cf. [Lemma 1(c), P. 51], we have that $Qx_0, \dots, Q^{i+1}x_0$ span the same subspace as $d_0, \dots, d_i$. Q.E.D.

From above discussion, we can also show that the stepsize parameters $\alpha_0, \dots, \alpha_{i-1}$ are nonzero; cf. [Lemma 4, P. 52]. Indeed, by the definition of g_0, we have $a_{00} = 1$. From Eq. (17), we have that, for $j = 1, \dots, i$, $a_{jj} = \alpha_{j-1} b_{(j-1)(j-1)}$. Since A is invertible and upper triangular, we have that $\alpha_{j-1} b_{(j-1)(j-1)} \neq 0$, which implies that $\alpha_{j-1} \neq 0$, $j = 1 \dots, i$.

P. 59 We provide a detailed derivation of quasi-Newton update formulas, assuming the related matrix inversions exist. We start with the DFP formula, which directly computes the approximation of the inverse of the Hessian. In particular, DFP computes the positive definite matrix D_{k+1} such that

$$D_{k+1}q_k = p_k, \tag{18}$$

where $p_k = x_{k+1} - x_k$ and $q_k = \nabla f(x_{k+1}) - \nabla f(x_k)$. By contrast, BFGS formula is derived by computing directly the approximation of Hessian H_{k+1} such that

$$q_k = H_{k+1}p_k.$$

The corresponding matrix D_{k+1} in BFGS is obtained via inversion of H_{k+1}.

For the DFP formula, since D_{k+1} is symmetric, we consider the candidate that takes the form

$$D_{k+1} = D_k + \alpha_k p_k p_k' + \beta_k D_k q_k q_k' D_k.$$

Multiplying on both sides q_k, we obtain

$$D_{k+1}q_k = D_k q_k + \alpha_k p_k p_k' q_k + \beta_k D_k q_k q_k' D_k q_k.$$

We obtain the equality (18) by setting α_k and β_k as

$$\alpha_k = \frac{1}{p_k' q_k}, \qquad \beta_k = -\frac{1}{q_k' D_k q_k}.$$

As a result, we obtain the DFP update equation

$$D_{k+1} = D_k + \frac{p_k p_k'}{p_k' q_k} - \frac{D_k q_k q_k' D_k}{q_k' D_k q_k}. \tag{19}$$

Repeating the computation with H_k, H_{k+1} in place of D_k, D_{k+1}, p_k and q_k in place of q_k and p_k, we obtain the following equation:

$$H_{k+1} = H_k + \frac{q_k q_k'}{p_k' q_k} - \frac{H_k p_k p_k' H_k}{p_k' H_k p_k}.$$

We will compute the inversion of the matrix H_{k+1}, using the Sherman-Morrison formula, on which we provide a brief discussion.

Lemma (1, P. 59)**.** *Let A be an n by n invertible matrix and $u, v \in \Re^n$ be some vectors. The matrix $A + uv'$ is invertible if and only if $1 + v'A^{-1}u \neq 0$.*

Proof. The following matrix identity holds:

$$\begin{bmatrix} I & 0 \\ v' & 1 \end{bmatrix} \begin{bmatrix} I + A^{-1}uv' & A^{-1}u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ -v' & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1}u \\ 0 & 1 + v'A^{-1}u \end{bmatrix},$$

where I is the identity matrix of suitable dimension. Taking determinant on both sides, we have that the determinant of the matrix $I + A^{-1}uv'$ equals to $1 + v'A^{-1}u$. Since $I + A^{-1}uv' = A^{-1}(A + uv')$, the determinant of $A + uv'$ is nonzero if and only if $1 + v'A^{-1}u \neq 0$. Q.E.D.

The following is the Sherman-Morrison formula. Its proof can be obtained by verifying the matrix identity.

Lemma (2, P. 59)**.** *Let A be an n by n invertible matrix and $u, v \in \Re^n$ be some vectors. If $1 + v'A^{-1}u \neq 0$, we have*

$$(A + uv')^{-1} = A^{-1} - \frac{A^{-1}uv'A^{-1}}{1 + v'A^{-1}u}. \tag{20}$$

Let us return to the derivation of BFGS. We have

$$H_{k+1} = H_k + \gamma_k q_k q_k' + \kappa_k H_k p_k p_k' H_k, \tag{21}$$

where

$$\gamma_k = \frac{1}{p_k' q_k}, \qquad \kappa_k = -\frac{1}{p_k' H_k p_k}.$$

We will apply Eq. (20) twice, first by treating the matrix $H_k + \gamma_k q_k q_k'$ as the matrix A, and then compute the inversion of the matrix $H_k + \gamma_k q_k q_k'$ by treating H_k as A. Note that we cannot treat $H_k + \kappa_k H_k p_k p_k' H_k$ as the matrix A in Eq. (20). This is because $1 + \kappa_k p_k' H_k (H_k^{-1}) H_k p_k = 0$, and therefore it is not invertible; cf. [Lemma 1, P. 59].

Applying Eq. (20) to (21) with the matrix $H_k + \gamma_k q_k q_k'$ in place of A, we have

$$H_{k+1}^{-1} = (H_k + \gamma_k q_k q_k')^{-1} - \frac{(H_k + \gamma_k q_k q_k')^{-1} \kappa_k H_k p_k p_k' H_k (H_k + \gamma_k q_k q_k')^{-1}}{1 + \kappa_k p_k' H_k (H_k + \gamma_k q_k q_k')^{-1} H_k p_k}. \tag{22}$$

Similarly, applying Eq. (20) to $(H_k + \gamma_k q_k q_k')^{-1}$, we have

$$(H_k + \gamma_k q_k q_k')^{-1} = H_k^{-1} - \frac{H_k^{-1} \gamma_k q_k q_k' H_k^{-1}}{1 + \gamma_k q_k' H_k^{-1} q_k}. \tag{23}$$

Next, we address the fraction term in (22). Applying Eq. (23) to the denominator in Eq. (22), we obtain

$$\begin{aligned}
&1 + \kappa_k p_k' H_k (H_k + \gamma_k q_k q_k')^{-1} H_k p_k \\
=&1 + \kappa_k p_k' H_k \left(H_k^{-1} - \frac{H_k^{-1} \gamma_k q_k q_k' H_k^{-1}}{1 + \gamma_k q_k' H_k^{-1} q_k} \right) H_k p_k \\
=&1 + \kappa_k p_k' H_k p_k - \frac{\kappa_k p_k' \gamma_k q_k q_k' p_k}{1 + \gamma_k q_k' H_k^{-1} q_k} \\
=&- \frac{\kappa_k p_k' q_k}{1 + \gamma_k q_k' H_k^{-1} q_k},
\end{aligned}$$

in view of the values of γ_k and κ_k.

We next simplify the term $(H_k + \gamma_k q_k q_k')^{-1} H_k p_k$ appeared in the numerator in Eq. (22), which reads

$$
\begin{aligned}
&(H_k + \gamma_k q_k q_k')^{-1} H_k p_k \\
=&\left(H_k^{-1} - \frac{H_k^{-1} \gamma_k q_k q_k' H_k^{-1}}{1 + \gamma_k q_k' H_k^{-1} q_k} \right) H_k p_k \\
=&p_k - \frac{H_k^{-1} \gamma_k q_k q_k' p_k}{1 + \gamma_k q_k' H_k^{-1} q_k} \\
=&p_k - \frac{H_k^{-1} q_k}{1 + \gamma_k q_k' H_k^{-1} q_k} \\
=&\frac{h_k}{1 + \gamma_k q_k' H_k^{-1} q_k},
\end{aligned}
$$

where $h_k = (1 + \gamma_k q_k' H_k^{-1} q_k) p_k - H_k^{-1} q_k$. Together with the preceding calculation, we can simplify the fractional term in Eq. (22) as follows:

$$
\begin{aligned}
&\frac{(H_k + \gamma_k q_k q_k')^{-1} \kappa_k H_k p_k p_k' H_k (H_k + \gamma_k q_k q_k')^{-1}}{1 + \kappa_k p_k' H_k (H_k + \gamma_k q_k q_k')^{-1} H_k p_k} \\
=&\frac{\kappa_k h_k h_k'}{(1 + \gamma_k q_k' H_k^{-1} q_k)^2} \frac{1 + \gamma_k q_k' H_k^{-1} q_k}{-\kappa_k p_k' q_k} \\
=& - \frac{h_k h_k'}{p_k' q_k (1 + \gamma_k q_k' H_k^{-1} q_k)} \\
=& - \frac{\big((1 + \gamma_k q_k' H_k^{-1} q_k) p_k - H_k^{-1} q_k\big)\big((1 + \gamma_k q_k' H_k^{-1} q_k) p_k - H_k^{-1} q_k\big)'}{p_k' q_k (1 + \gamma_k q_k' H_k^{-1} q_k)}.
\end{aligned}
$$

Applying the preceding equation and Eq. (23) into (22) and by setting $D_{k+1} = H_{k+1}^{-1}$ and $D_k = H_k^{-1}$, we have

$$
\begin{aligned}
&D_{k+1} \\
=&D_k - \frac{D_k \gamma_k q_k q_k' D_k}{1 + \gamma_k q_k' D_k q_k} + \frac{\big((1 + \gamma_k q_k' D_k q_k) p_k - D_k q_k\big)\big((1 + \gamma_k q_k' D_k q_k) p_k - D_k q_k\big)'}{p_k' q_k (1 + \gamma_k q_k' D_k q_k)} \\
=&D_k + \frac{-D_k q_k q_k' D_k + \big((1 + \gamma_k q_k' D_k q_k) p_k - D_k q_k\big)\big((1 + \gamma_k q_k' D_k q_k) p_k - D_k q_k\big)'}{p_k' q_k (1 + \gamma_k q_k' D_k q_k)} \\
=&D_k + \frac{1}{p_k' q_k (1 + \gamma_k q_k' D_k q_k)} \big(- D_k q_k q_k' D_k + (1 + \gamma_k q_k' D_k q_k)^2 p_k p_k' - \\
&(1 + \gamma_k q_k' D_k q_k) D_k q_k p_k' - (1 + \gamma_k q_k' D_k q_k) p_k q_k' D_k + D_k q_k q_k' D_k \big) \\
=&D_k + \frac{(1 + \gamma_k q_k' D_k q_k)^2 p_k p_k' - (1 + \gamma_k q_k' D_k q_k) D_k q_k p_k' - (1 + \gamma_k q_k' D_k q_k) p_k q_k' D_k}{p_k' q_k (1 + \gamma_k q_k' D_k q_k)} \\
=&D_k + \frac{(1 + \gamma_k q_k' D_k q_k) p_k p_k'}{p_k' q_k} - \frac{D_k q_k p_k' + p_k q_k' D_k}{p_k' q_k}.
\end{aligned}
$$

In view of $\gamma_k = 1/p_k'q_k$, we have

$$D_{k+1} = D_k + \frac{p_k p_k'}{p_k' q_k} + \frac{(q_k' D_k q_k) p_k p_k'}{(p_k' q_k)^2} - \frac{D_k q_k p_k' + p_k q_k' D_k}{p_k' q_k} \tag{24}$$

Equation (24) is the BFGS update given in [NLP Ex2.2.2, P. 145].

To obtain [COLMM Eq. (80), P. 59], we focus on the term $\frac{(q_k' D_k q_k) p_k p_k'}{(p_k' q_k)^2} - \frac{D_k q_k p_k' + p_k q_k' D_k}{p_k' q_k}$ by forming a quadratic form as

$$\begin{aligned}
&\frac{(q_k' D_k q_k) p_k p_k'}{(p_k' q_k)^2} - \frac{D_k q_k p_k' + p_k q_k' D_k}{p_k' q_k} \\
=&(q_k' D_k q_k)\left(\frac{p_k p_k'}{(p_k' q_k)^2} - \frac{D_k q_k p_k' + p_k q_k' D_k}{(q_k' D_k q_k)(p_k' q_k)}\right) \\
=&(q_k' D_k q_k)\left(\frac{p_k p_k'}{(p_k' q_k)^2} - \frac{D_k q_k p_k'}{(q_k' D_k q_k)(p_k' q_k)} - \frac{p_k q_k' D_k}{(q_k' D_k q_k)(p_k' q_k)} + \frac{D_k q_k q_k' D_k}{(q_k' D_k q_k)^2} - \frac{D_k q_k q_k' D_k}{(q_k' D_k q_k)^2}\right) \\
=&-\frac{D_k q_k q_k' D_k}{q_k' D_k q_k} + (q_k' D_k q_k)\left(\frac{p_k}{p_k' q_k} - \frac{D_k q_k}{q_k' D_k q_k}\right)\left(\frac{p_k}{p_k' q_k} - \frac{D_k q_k}{q_k' D_k q_k}\right)'.
\end{aligned}$$

Therefore, we obtain the BFGS formula

$$D_{k+1} = D_k + \frac{p_k p_k'}{p_k' q_k} - \frac{D_k q_k q_k' D_k}{q_k' D_k q_k} + (q_k' D_k q_k)\left(\frac{p_k}{p_k' q_k} - \frac{D_k q_k}{q_k' D_k q_k}\right)\left(\frac{p_k}{p_k' q_k} - \frac{D_k q_k}{q_k' D_k q_k}\right)'.$$

Together with the DFP update given in Eq. (19), the class of updating formula can be written as

$$D_{k+1} = D_k + \frac{p_k p_k'}{p_k' q_k} - \frac{D_k q_k q_k' D_k}{q_k' D_k q_k} + \zeta_k (q_k' D_k q_k)\left(\frac{p_k}{p_k' q_k} - \frac{D_k q_k}{q_k' D_k q_k}\right)\left(\frac{p_k}{p_k' q_k} - \frac{D_k q_k}{q_k' D_k q_k}\right)', \tag{25}$$

where $0 \le \zeta_k \le 1$.

Alternatively, starting with Eq. (24) we can also group together the terms $D_k + \frac{(q_k' D_k q_k) p_k p_k'}{(p_k' q_k)^2} - \frac{D_k q_k p_k' + p_k q_k' D_k}{p_k' q_k}$ by using the following identity

$$(q_k' D_k q_k) p_k p_k' = p_k (q_k' D_k q_k) p_k' = p_k q_k' D_k q_k p_k' = (q_k p_k')' D_k (q_k p_k'),$$

as $q_k' D_k q_k$ is a scalar. Together with $\gamma_k = 1/(p_k' q_k)$, we obtain

$$\begin{aligned}
&D_k + \frac{(q_k' D_k q_k) p_k p_k'}{(p_k' q_k)^2} - \frac{D_k q_k p_k' + p_k q_k' D_k}{p_k' q_k} \\
=&D_k + \gamma_k^2 (q_k p_k')' D_k (q_k p_k') - \gamma_k D_k (q_k p_k') - \gamma_k (q_k p_k')' D_k \\
=&D_k\big(I - \gamma_k (q_k p_k')\big) + \gamma_k (q_k p_k')' D_k \big(\gamma (q_k p_k') - I\big) \\
=&\big(I - \gamma_k (q_k p_k')'\big) D_k \big(I - \gamma (q_k p_k')\big) \\
=&(I - \gamma_k q_k p_k')' D_k (I - \gamma_k q_k p_k').
\end{aligned}$$

In view of Eq. (24), we obtain

$$D_{k+1} = \left(I - \frac{q_k p_k'}{p_k' q_k}\right)' D_k \left(I - \frac{q_k p_k'}{p_k' q_k}\right) + \frac{p_k p_k'}{p_k' q_k}.$$

This formula is used for the derivation of L-BFGS; see [NLP Ex2.2.3, P. 145].

P. 60 It is implicitly required here that $\alpha_k \geq 0$, and it has been stated in the first paragraph in [COLMM Section 1.3.5, P. 59]. The proof also uses this condition. Indeed, from $\nabla f(x_k)'d \neq \nabla f(x_{k+1})'d$, we have $x_k \neq x_{k+1}$, which implies that $\alpha_k \neq 0$. Together with $\alpha_k \geq 0$, we have $\alpha_k > 0$. In addition, $p_k' q_k = \alpha_k d_k' [\nabla f(x_{k+1} - \nabla f(x_k)]$. Together with $\alpha_k > 0$, the condition $\nabla f(x_k)' d_k < \nabla f(x_{k+1})' d_k$ implies that $p_k' q_k > 0$.

P. 61

Lemma (1, P. 61)**.** *Consider the quasi-Newton algorithm applied to minimization of the positive definite quadratic function $f(x) = \frac{1}{2}x'Qx$, where Q is positive definite. Suppose that D_k is positive definite, $\nabla f(x_k) \neq 0$, and the stepsize α_k is chosen by*

$$f(x_k + \alpha_k x_k) = \min_{\alpha} f(x_k + \alpha x_k). \tag{26}$$

Then $\alpha_k > 0$, and D_{k+1} given by Eq. (25) is well-defined and positive definite.

Note that although α_k selected according to Eq. (26) satisfies the condition $\nabla f(x_k)' d_k < \nabla f(x_{k+1})' d_k$ used in [COLMM Prop. 1.20, P. 60], we cannot apply [COLMM Prop. 1.20, P. 60] here to assert that D_{k+1} is positive definite. This is because that it is implicitly required in [COLMM Prop. 1.20, P. 60] that $\alpha_k \geq 0$, and the proof of [COLMM Prop. 1.20, P. 60] also relies on this condition; see [P. 60]. Yet, $\alpha_k \geq 0$ is something we aim to show here.

Proof. Since $\nabla f(x_k) \neq 0$, we have $x_k \neq 0$. Since α_k is defined according to Eq. (26), we have $(x_k + \alpha_k d_k)' Q d_k = 0$. Expanding the term $d_k = -D_k Q x_k$, we obtain

$$\alpha_k x_k' Q D_k Q D_k x_k = x_k' Q D_k Q x_k.$$

Therefore, we have $\alpha_k > 0$. This implies that $q_k = \alpha_k Q d_k \neq 0$. In view that $p_k = \alpha_k d_k$, we have $q_k' p_k = \alpha_k^2 d_k Q d_k > 0$. Therefore, D_{k+1} given by Eq. (25) is well-defined. The proof for D_{k+1} being positive definite is identical to the corresponding part in the proof of [COLMM Prop. 1.20, P. 60]. Q.E.D.

Since none of the vectors $x_0, \ldots, x_{n-1}$ is optimal, repeatedly applying [Lemma 1, P. 61], we have that starting from a positive definite matrix D_0, the stepsizes $\alpha_0, \ldots, \alpha_{n-1}$ are positive, and the matrices $D_1, \ldots, D_n$ are positive definite. Moreover, we have $p_0 \ldots, p_{n-1}$ are nonzero, as $\alpha_0, \ldots, \alpha_{n-1}$ are positive.

In what follows, we provide somewhat more direct arguments to show that

$$d_i' Q d_j = 0, \qquad 0 \leq i < j \leq k,\ k = 1, \ldots, n-1 \tag{27}$$

$$D_{k+1} q_i = p_i, \qquad 0 \leq i \leq k,\ k = 0, \ldots, n-1 \tag{28}$$

which follows closely those given in the proof for [COLMM Prop. 1.21, P. 61]. First, for $k = 0, \dots, n-1$, we have $D_{k+1}q_k = p_k$ as shown in [COLMM P. 61]. In view that $p_k = \alpha_k d_k$ and $Qp_k = q_k$, we have

$$D_{k+1}Qd_k = d_k, \qquad k = 0, \dots, n-1. \tag{29}$$

Therefore, for $k = 1, \dots, n-1$, we have

$$d_k' Q d_{k-1} = -x_k Q D_k Q d_{k-1} = -x_k Q d_{k-1} = 0,$$

where the second to last equality is due to Eq. (29) and the last equality is due to that $\nabla f(x_k)' d_{k-1} = 0$. Moreover, the above two derivations yield

$$p_k' Q p_{k-1} = p_k' q_{k-1} = q_k' p_{k-1} = q_k' D_k q_{k-1} = 0, \quad k = 1, \dots, n-1. \tag{30}$$

The above arguments have shown that Eq. (27) holds for $k = 1$ and Eq. (28) holds for $k = 0$. To show that Eq. (28) holds for $k = 1$, we only need to show that $D_2 q_0 = p_0$. This can be shown by using Eq. (30) with $k = 1$.

Then we can continue with induction, assuming that Eqs. (27) and (28) hold for k, where $k \leq n-2$. From induction hypothesis, we have $D_{k+1}Qd_i = D_{k+1}Qp_i/\alpha_i = p_i/\alpha_i = d_i$, $0 \leq i \leq k$. Therefore,

$$d_{k+1}' Q d_i = -x_{k+1}' Q D_{k+1} Q d_i = -x_{k+1} Q d_i = -(x_{i+1} + \alpha_{i+1} d_{i+1} + \cdots + \alpha_k d_k) Q d_i.$$

By induction hypothesis, we have $(\alpha_{i+1} d_{i+1} + \cdots + \alpha_k d_k) Q d_i = 0$. Moreover, $x_{i+1} Q d_i$ due to α_i is computed via minimizing rule. Therefore, we show Eq. (27) for $k+1$. Consequently, we also have

$$p_{k+1}' Q p_i = p_{k+1}' q_i = q_{k+1}' p_i = q_{k+1}' D_{k+1} q_i = 0, \quad 0 \leq i \leq k, \tag{31}$$

where in the third equality, we used induction hypothesis $D_{k+1} q_i = p_i$. Then $D_{k+2} q_i = p_i$ can be shown by using the equalities given in Eq. (31), as is given in [COLMM P. 62].

As the last step of derivation, we have $D_n Q p_i = p_i$, $i = 0, \dots, n-1$, and the vectors $p_0, \dots, p_{n-1}$ are Q-conjugate, thus linearly independent. This implies that $D_n Q v = v$ for all $v \in \Re^n$. Applying following lemma, we have $D_n = Q^{-1}$.

Lemma (2, P. 61)**.** *Let A be an n by n matrix such that $Av = v$ for all $v \in \Re^n$. Then A is the identity matrix.*

Proof. Since $Av = v$, then $(I - A)v = 0$ for all v. We can consider $v = e_i$, where e_i is the vector that has ith component being 1 and 0 otherwise. Then $(I - A)e_i = 0$ shows that the ith column of A is e_i. This concludes the proof. Q.E.D.

P. 62

Lemma (1, P. 62)**.** *Suppose that the vectors $d_0, \ldots, d_k \in \Re^n$ are linearly independent and the n by n matrix Q is positive definite. Then the matrix R is positive definite, where*

$$R = \begin{bmatrix} d_0'Qd_0 & d_1'Qd_0 & \cdots & d_k'Qd_0 \\ d_1'Qd_0 & d_1'Qd_1 & \cdots & d_k'Qd_1 \\ \vdots & \vdots & \ddots & \vdots \\ d_k'Qd_0 & d_k'Qd_1 & \cdots & d_k'Qd_k \end{bmatrix}.$$

Proof. Consider the vector $a = [\alpha_0 \; \alpha_1 \; \ldots \; \alpha_i]'$. We have

$$a'Ra = \sum_{i=0}^{k}\sum_{j=0}^{k} \alpha_i \alpha_j d_i' Q d_j = \left(\sum_{i=0}^{k} \alpha_i d_i\right)' Q \left(\sum_{i=0}^{k} \alpha_i d_i\right).$$

However, since $d_0, \ldots, d_k \in \Re^n$ are linearly independent and Q is positive definite, $(\sum_{i=0}^{k} \alpha_i d_i)' Q (\sum_{i=0}^{k} \alpha_i d_i) = 0$ if and only if $\alpha_0 = \alpha_1 = \cdots = \alpha_k = 0$ and positive otherwise. Therefore, matrix R is positive definite. Q.E.D.

Lemma (2, P. 62)**.** *Suppose that the vectors $d_0, \ldots, d_k \in \Re^n$ are linearly independent, $x_0 \in \Re^n$ is some vector, and Q is an n by n positive definite matrix. Then the function $f(x) = \frac{1}{2}x'Qx$ admits a unique minimizer on the manifold M defined as*

$$M = \{z \,|\, z = x_0 + \alpha_0 d_0 + \alpha_1 d_1 + \cdots + \alpha_k d_k, \; \alpha_0, \ldots, \alpha_k \in \Re\}.$$

Proof. First, we show that f attains a minimum in M. Indeed, we have

$$\inf_{z \in M} f(z) = \inf_{z \in C} f(z),$$

where $C = M \cap \{z \,|\, f(z) \leq f(x_0)\}$. Since the set C is compact, f attains a minimum on C, which also belongs to M.

Suppose that $z^* = x_0 + \sum_{i=0}^{k} \alpha_i^* d_i$ attains the minimum. Then it satisfies the first order necessary condition, which states

$$\begin{aligned} d_0'Qd_0\alpha_0^* + d_1'Qd_0\alpha_1^* + \cdots + d_k'Qd_0\alpha_k^* &= -d_0'Qx_0 \\ d_1'Qd_0\alpha_0^* + d_1'Qd_1\alpha_1^* + \cdots + d_k'Qd_1\alpha_k^* &= -d_1'Qx_0 \\ &\vdots \\ d_k'Qd_0\alpha_0^* + d_k'Qd_1\alpha_1^* + \cdots + d_k'Qd_k\alpha_k^* &= -d_k'Qx_0. \end{aligned}$$

The above equation can be written in a compact form as $Ra^* = b$, where $a^* = [\alpha_0^* \; \cdots \; \alpha_k^*]'$ and $b = [-d_0'Qx_0 \; -d_1'Qx_0 \; \cdots \; -d_k'Qx_0]'$, and R is given as in [Lemma 1, P. 62]. Since by [Lemma 1, P. 62], R is positive definite, thus invertible. The necessary condition uniquely define a^*. Q.E.D.

Lemma (3, P. 62). *Starting with $D_0 = I$, let $x_0 \ldots, x_k$ and $d_0, \ldots, d_k$ be the sequences generated by the quasi-Newton algorithm applied to minimization of the positive definite quadratic function $f(x) = \frac{1}{2}x'Qx$, where Q is positive definite. Suppose that the stepsize α_i is chosen by*

$$f(x_i + \alpha_k x_i) = \min_{\alpha} f(x_i + \alpha x_i).$$

Assume further that $x_0, \ldots, x_k$ are not optimal.

(a) *For $i = 0, \ldots, k$, there exists scalars $\beta^i_{\ell j}$ and b^i_ℓ such that*

$$D_i = I + \sum_{\ell=0}^{i} \sum_{j=0}^{i} \beta^i_{\ell j} \nabla f(x_\ell) \nabla f(x_j)', \tag{32}$$

$$d_i = \sum_{\ell=0}^{i} b^i_\ell \nabla f(x_\ell). \tag{33}$$

(b) *For $i = 0, \ldots, k$, we have $M_i = \tilde{M}_i$, where*

$$M_i = \{z \mid z = x_0 + \gamma_0 \nabla f(x_0) + \cdots + \gamma_i \nabla f(x_i),\ \gamma_0, \ldots, \gamma_i \in \Re\},$$
$$\tilde{M}_i = \{z \mid z = x_0 + \gamma_0 d_0 + \cdots + \gamma_i d_i,\ \gamma_0, \ldots, \gamma_i \in \Re\}.$$

(c) *Starting with $\hat{x}_0 = x_0$, let $\hat{x}_0 \ldots, \hat{x}_k$ be the sequences generated by the conjugate gradient methods. We have $\hat{x}_i = x_i$, $i = 0, \ldots, k$.*

Proof. (a) Since $x_0, \ldots, x_k$ are not optimal, then $D_1, \ldots, D_k$ are well-defined according to [Lemma 1, P. 61]. We show Eqs. (32) and (33) together by induction. Clearly they hold for $i = 0$ since $D_0 = I$ and $d_0 = -D_0 \nabla f(x_0) = -\nabla f(x_0)$. Suppose that they hold for i. Then to show Eq. (32) holds for D_{i+1}, we check the terms p_i and $D_i q_i$ appeared in Eq. (25). Clearly, $p_i = \alpha_i d_i$, where $d_i = \sum_{\ell=0}^{i} b^i_\ell \nabla f(x_\ell)$, according to induction hypothesis. Similarly, $D_i q_i = (I + \sum_{\ell=0}^{i} \sum_{j=0}^{i} \beta^i_{\ell j} \nabla f(x_\ell) \nabla f(x_j)')[\nabla f(x_{i+1}) - \nabla f(x_i)]$. Therefore, D_{i+1} can be written as the form given by Eq. (32). Since $d_{i+1} = -D_{i+1} \nabla f(x_{i+1})$. we have Eq. (33) holds for $i + 1$.

(b) From part (a), we have that d_ℓ, $\ell = 0, \ldots, i$, are linear combinations of $\nabla f(x_\ell)$, $\ell = 0, \ldots, i$. From [COLMM Prop. 1.21(a), P. 61], the vectors d_ℓ, $\ell = 0, \ldots, i$ are linearly independent. Then by [Lemma 1, P. 54], the vectors $\nabla f(x_\ell)$, $\ell = 0, \ldots, i$ are also linearly independent, and span the same space as d_ℓ, $\ell = 0, \ldots, i$.

(c) We will show that the vectors $\hat{x}_0 \ldots, \hat{x}_k$ are well-defined and $\hat{x}_i = x_i$, $i = 0, \ldots, k$, simultaneously by induction. To this end, denote by $\hat{d}_0 \ldots, \hat{d}_k$ the Q-conjugate vectors computed via the conjugate gradient method. In addition, we define the manifolds $\hat{M}_i$ as

$$\hat{M}_i = \{z \mid z = \hat{x}_0 + \gamma_0 \hat{d}_0 + \cdots + \gamma_i \hat{d}_i,\ \gamma_0, \ldots, \gamma_i \in \Re\}.$$

If $\hat{x}_0, \ldots, \hat{x}_i$ are nonoptimal, by [Lemma 3, P. 52], $\nabla f(\hat{x}_0), \ldots, \nabla f(\hat{x}_i)$ are linearly independent. Moreover, by [Lemma 1, P. 51], we have

$$\hat{M}_i = \{z \,|\, z = \hat{x}_0 + \gamma_0 \nabla f(\hat{x}_0) + \cdots + \gamma_i \nabla f(\hat{x}_i),\ \gamma_0, \ldots, \gamma_i \in \Re\}.$$

We will show by induction that $\hat{M}_i = M_i$. Then by [Lemma 2, P. 62], $\hat{x}_{i+1} = x_{i+1}$.

Since we have $\hat{x}_0 = x_0$, $\hat{d}_0 = d_0$ is nonzero, and $\hat{M}_0 = M_0$. As a result, we have $\hat{x}_1 = x_1$. Moreover, $\hat{d}_1$ is well-defined and is nonzero. Suppose that $\hat{x}_j = x_j$, $j = 0, \ldots, i$, where $i \leq k-1$. Therefore, $\hat{d}_i$ is well-defined and nonzero, and $\hat{M}_i = M_i$. By [Lemma 2, P. 62], $\hat{x}_{i+1} = x_{i+1}$. Q.E.D.

P. 63 Conjugate gradient method computes $d_k = -g_k + \beta_k d_{k-1}$, where $\beta_k = |g_k|^2/|g_{k-1}|^2$ requires $2n+1$ multiplications.

P. 64 Considering computing D_{k+1} via Eq. (25). Computing $p_k p_k'$ requires n^2 multiplications, and $p_k' q_k$ n multiplications. Computing first $D_k q_k$ requires n^2 multiplications, then computing $D_k q_k q_k' D_k$ requires another n^2 multiplications. Proceeding similarly, we can see that computing D_{k+1} requires no more than Mn^2 multiplications, where M is some constant.

P. 64 In Newton's method, once Cholesky factorization for $\nabla^2 f(x_k)$ is obtained, i.e., $\nabla^2 f(x_k) = L_k L_k'$, computing d_k via solving the equation $L_k L_k' d_k = -\nabla f(x_k)$ requires $n^2 + n$ multiplication. To see this, we first solve $L_k' d_k$, which requires $n(n+1)/2$ multiplications. Then we solve d_k, which also requires $n(n+1)/2$ multiplications.

P.65 We first quantify the approximation error of forward difference formula for computing $\partial f(x)/\partial x^i$, assuming $f \in C^2$. First, we note that $\partial f(x)/\partial x^i = \nabla f(x)' e_i$, where e_i is the ith column of the identity matrix. By mean value theorem, we have

$$f(x + he_i) = f(x) + h\nabla f(x)' e_i + \frac{1}{2} h^2 e_i' \nabla^2 f\big(x + \alpha(x, x + he_i) he_i\big) e_i, \quad (34)$$

where $\alpha(x, x + he_i) \in [0, 1]$, which depends on x and $x + he_i$. Then we have that

$$(1/h)[f(x + he_i) - f(x)] - \partial f(x)/\partial x^i = \frac{1}{2} hg\big(x + \alpha(x, x + he_i) he_i\big),$$

where $g(x) = \partial^2 f(x)/\partial x^i \partial x^i$. Let $\{h_\ell\} \subset \Re$ be a sequence such that $h_\ell \to 0$ as $\ell \to \infty$. Then the sequence $\{q_\ell\}$ is bounded by some value M, where $q_\ell = g\big(x + \alpha(x, x + h_\ell e_i) h_\ell e_i\big)$.

Next, we quantify the approximation error of the central difference formula, assuming $f \in C^3$. By mean value theorem, we also have

$$f(x - he_i) = f(x) - h\nabla f(x)' e_i + \frac{1}{2} h^2 e_i' \nabla^2 f\big(x - \alpha(x, x - he_i) he_i\big) e_i, \quad (35)$$

where $\alpha(x, x - he_i) \in [0, 1]$. Subtracting Eq. (34) from Eq. (35), we have

$$\begin{aligned}&(1/2h)[f(x + he_i) - f(x - he_i)] - \partial f(x)/\partial x^i\\=&\frac{1}{4}h\big[g\big(x + \alpha(x, x + he_i)he_i\big) - g\big(x - \alpha(x, x - he_i)he_i\big)\big].\end{aligned}$$

Since $f \in C^3$, applying mean value theorem, we have

$$\begin{aligned}&g\big(x + \alpha(x, x + he_i)he_i\big)\\=&g(x) + \nabla g\Big(x + \beta(x, x + \alpha(x, x + he_i)he_i)\alpha(x, x + he_i)he_i\Big)'\alpha(x, x + he_i)he_i,\\&g\big(x - \alpha(x, x - he_i)he_i\big)\\=&g(x) - \nabla g\Big(x - \beta(x, x - \alpha(x, x - he_i)he_i)\alpha(x, x - he_i)he_i\Big)'\alpha(x, x - he_i)he_i,\end{aligned}$$

where $\beta(x, x+\alpha(x, x+he_i)he_i) \in [0, 1]$ and $\beta(x, x-\alpha(x, x-he_i)he_i) \in [0, 1]$ are the coefficients appeared in the mean value theorem. Therefore, we have

$$\begin{aligned}&g\big(x + \alpha(x, x + he_i)he_i\big) - g\big(x - \alpha(x, x + he_i)he_i\big)\\=&\Big[\nabla g\Big(x + \beta(x, x + \alpha(x, x + he_i)he_i)\alpha(x, x + he_i)he_i\Big)'\alpha(x, x + he_i)e_i+\\&\nabla g\Big(x - \beta(x, x - \alpha(x, x - he_i)he_i)\alpha(x, x - he_i)he_i\Big)'\alpha(x, x - he_i)e_i\Big]h.\end{aligned}$$

Let $\{h_\ell\} \subset \Re$ be a sequence such that $h_\ell \to 0$ as $\ell \to \infty$. Then the sequence $\{r_\ell\}$ is bounded by some value M, where

$$\begin{aligned}r_\ell =&\nabla g\Big(x + \beta(x, x + \alpha(x, x + h_\ell e_i)h_\ell e_i)\alpha(x, x + h_\ell e_i)h_\ell e_i\Big)'\alpha(x, x + h_\ell e_i)e_i+\\&\nabla g\Big(x - \beta(x, x - \alpha(x, x - h_\ell e_i)h_\ell e_i)\alpha(x, x - h_\ell e_i)h_\ell e_i\Big)'\alpha(x, x - h_\ell e_i)e_i.\end{aligned}$$

P. 68 Let us provide an alternative proof. We define the set $X = \{x \mid |x| = 1,\ x'Px \le 0\}$. If $X = \emptyset$, then $c = 0$ satisfies $P + cQ > 0$. Otherwise, for every $x \in X$, we have $x'Qx > 0$ since $x'Px > 0$ for all x satisfying $x'Qx = 0$. Consider the function $f : X \mapsto \Re$ defined as $f(x) = -x'Px/(x'Qx)$, which is continuous and nonnegative. Since the set X is compact, we have $\max_{x\in X} f(x)$ is attained and finite. Let $c > \max_{x\in X} f(x)$. We have $P + cQ > 0$. From this proof, it can be seen that if $P + \bar{c}Q > 0$ for some $\bar{c}$, then $P + cQ > 0$ for all $c \ge \bar{c}$.

P. 68

Lemma (1, P. 68)**.** *Let* $\{a_k\}$ *and* $\{b_k\}$ *be real sequences such that* $\liminf_k a_k = a \in \Re$ *and* $\limsup_k b_k = b \in \Re$. *Then* $\{c_k\}$ *and* $\{d_k\}$ *are real sequences, where* $c_k = \inf_{n\ge k} a_n$ *and* $d_k = \sup_{n\ge k} b_n$.

Proof. We prove the case for d_k. Since $d_k \to b$, for some $\epsilon > 0$, there exists a $\bar{k}$ such that $|d_k - b| < \epsilon$ for all $k \ge \bar{k}$. Therefore, what remains to show is that $d_k \in \Re$, $k = 1, 2, \ldots, \bar{k} - 1$. By the definition of d_k, we have $d_{\bar{k}} \le d_k$. Moreover, $d_k \le \max\{b_k, b_{\bar{k}-1}, d_{\bar{k}} + \epsilon\}$. Therefore, $d_k \in \Re$, $k = 1, \ldots, \bar{k} - 1$. Q.E.D.

Lemma (2, P. 68)**.** *Let* $\{b_k\}$ *be a real sequence* $\sup\{b_k\} = b \in \Re$*. Then for every* $c \in \Re$*, we have* $\sup\{b_k + c\} = b + c$*.*

Proof. For all k, we have $b_k + c \le b + c$ so that $b + c$ is an upper bound of $\{b_k + c\}$. Moreover, for every $\epsilon > 0$, there exists some k such that $b_k + c > b + c - \epsilon$. Therefore, $\sup\{b_k + c\} = b + c$. Q.E.D.

Lemma (3, P. 68)**.** *Let* $\{a_k\}$ *and* $\{b_k\}$ *be real sequences such that* $\lim a_k = a \in \Re$ *and* $\limsup b_k = b \in \Re$*. Then we have*

$$\limsup_{k\to\infty} (a_k + b_k) = \lim_{k\to\infty} a_k + \limsup_{k\to\infty} b_k.$$

Proof. We first show that $\limsup(a_k + b_k) \le \lim a_k + \limsup_{k\to\infty} b_k$. For every k, we have $\sup_{n\ge k}(a_n + b_n) \le \sup_{n\ge k} a_n + \sup_{n\ge k} b_n$. Taking limit on both sides, we have $\limsup(a_k + b_k) \le \lim_{k\to\infty}(\sup_{n\ge k} a_n + \sup_{n\ge k} b_n)$. By [Lemma 1, P. 68], the sequences $\{\sup_{n\ge k} a_n\}$ and $\{\sup_{n\ge k} b_n\}$ with index k are real sequences, which are also convergent. Therefore, we have $\limsup(a_k + b_k) \le \limsup a_k + \limsup b_k = \lim a_k + \limsup b_k$.

Conversely, for every k, we have

$$a_n + b_n \ge b_n + \inf_{\ell\ge k}(a_\ell), \qquad n \ge k.$$

Taking supreme on both sides and applying [Lemma 2, P. 68] with $c = \inf_{\ell\ge k} a_\ell$, we have

$$\sup_{n\ge k}(a_n + b_n) \ge \sup_{n\ge k}\left(b_n + \inf_{\ell\ge k} a_\ell\right) = \sup_{n\ge k} b_n + \inf_{n\ge k} a_n.$$

By [Lemma 1, P. 68], the sequences $\{\inf_{n\ge k} a_n\}$ and $\{\sup_{n\ge k} b_n\}$ with index k are real sequences, which are also convergent. Taking limits on both sides, we have

$$\limsup_{k\to\infty} (a_k + b_k) \ge \lim_{k\to\infty}\left(\sup_{n\ge k} b_n + \inf_{n\ge k} a_n\right) = \limsup_{k\to\infty} b_k + \lim_{k\to\infty} a_k,$$

which is the desired equality. Q.E.D.

From [Lemma 3, P. 68], we have

$$\lim \sup_{k\to\infty, k\in K} (x_k' P x_k + k x_k' Q x_k) = \bar{x}' P \bar{x} + \lim \sup_{k\to\infty, k\in K} (k x_k' Q x_k) \le 0. \quad (36)$$

We now claim that $\bar{x}' Q \bar{x} = 0$. Suppose otherwise, since $x_k' Q x_k \ge 0$, we must have $\bar{x}' Q \bar{x} > 0$. As a result, for every $k \in K$, $\sup_{n\ge k}(n x_n' Q x_n) = \infty$, which implies that $\limsup_{k\to\infty, k\in K}(k x_k' Q x_k) = \infty$. This contradicts with Eq. (36).

From $\bar{x}'Q\bar{x} = 0$, we have $\bar{x}'P\bar{x} > 0$ according to the conditions stated in [COLMM Lemma 1.25, P. 68]. Together with $\limsup_{k\to\infty, k\in K}(kx_k'Qx_k) \geq 0$, we have $\bar{x}'P\bar{x}+\limsup_{k\to\infty, k\in K}(kx_k'Qx_k) > 0$, which contradicts with Eq. (36).

P. 69 Note that from the discussion given before [COLMM Prop. 1.26, P. 69], we have that there exists some $\bar{c} > 0$ such that the function $L_{\bar{c}}(x, \lambda^*)$, as a function of x, satisfies the conditions stated in [COLMM Prop. 1.4, P. 19]. According to [COLMM Prop. 1.4, P. 19], there exists some $\gamma > 0$ and $\delta > 0$ such that

$$L_{\bar{c}}(x, \lambda^*) \geq L_{\bar{c}}(x^*, \lambda^*) + \gamma|x - x^*|^2, \quad \text{for all } x \in S(x^*; \delta).$$

Since $h(x^*) = 0$, we have for all $c \geq \bar{c}$, $L_c(x, \lambda^*) \geq L_{\bar{c}}(x, \lambda^*)$ for all $x \in S(x^*; \delta)$, and $L_c(x^*, \lambda^*) = L_{\bar{c}}(x^*, \lambda^*)$. This yields that

$$L_c(x, \lambda^*) \geq L_c(x^*, \lambda^*) + \gamma|x - x^*|^2, \quad \text{for all } x \in S(x^*; \delta) \text{ and } c \geq \bar{c}.$$

P. 70

Lemma (1, P. 70)**.** *Suppose that U is an open set of $\Re^r$. Let $A(u)$ and $B(u)$ represent n by n and n by m matrices that depend continuously on $u \in U$. Moreover, $A(u)$ is symmetric for all u. Let $\bar{u} \in U$ satisfies that*

$$z'A(\bar{u})z > 0 \quad \textit{for all } z \neq 0 \textit{ and } B(\bar{u})'z = 0. \tag{37}$$

Then there exists some $\delta > 0$ such that $S(\bar{u}; \delta) \subset U$ and for all $u \in S(\bar{u}; \delta)$, there holds

$$z'A(u)z > 0 \quad \textit{for all } z \neq 0 \textit{ and } B(u)'z = 0.$$

Proof. Since U is open, there exists some $\bar{\delta} > 0$ such that $S(\bar{u}; \bar{\delta}) \subset U$. Assuming for all $0 < \delta < \bar{\delta}$, the desired condition does not hold. For every k such that $1/k < \bar{\delta}$, there exists some $u_k \in S(\bar{u}; 1/k)$ such that for some z_k, we have $|z_k| = 1$, $z_k'A(u_k)z_k \leq 0$, and $B(u_k)'z_k = 0$. Clearly, u_k converges to $\bar{u}$. Given that $|z_k| = 1$ for all k, there exists some subsequence $\{z_k\}_K$ that is convergent, whose limit is denoted as $\bar{z}$. Since $A(u)$ and $B(u)$ depend continuous on u, taking limit over K, we have $\bar{z}'A(\bar{u})\bar{z} \leq 0$, $|\bar{z}| = 1$, and $B(\bar{u})'\bar{z} = 0$. This contradicts with the condition (37) of $\bar{u}$. Q.E.D.

P. 72 Consider the following optimization problem

$$\begin{aligned} \text{minimize} \quad & \bar{f}(x, z) \\ \text{subject to} \quad & \bar{h}_i(x, z) = 0, \; i = 1, \ldots, m, \\ & \bar{g}_j(x, z) = 0, \; j = 1, \ldots, r, \end{aligned} \tag{38}$$

where $\bar{f}$, $\bar{h}_i$, and $\bar{g}_j$ are the functions defined in [COLMM P. 72]. We show that the following holds.

Lemma (1, P. 72)**.** *Suppose that $g : \Re^n \mapsto \Re^r$ is continuous at the vector x^*. The vector x^* is the local minimum of (NLP) if and only if (x^*, z^*), where $z_j^* = [-g_j(x^*)]^{1/2}$, $j = 1, \ldots, r$, is the local minimum of problem* (38).

Proof. Denote $X = \{x \,|\, h(x) = 0,\, g(x) \le 0\}$, $\hat{X} = \{x \,|\, g(x) \le 0\}$, $\overline{X} = \{(x,z) \,|\, \bar{h}(x,z) = 0,\, \bar{g}(x,z) = 0\}$, and $v^* = (x^*, z^*)$. For every x such that $g(x) \le 0$, define $a : \hat{X} \mapsto \Re^r$, whose jth component, denoted as $a_j(x)$ is given by $a_j(x) = [-g_j(x)]^{1/2}$, $j = 1, \ldots, r$.

We first show the necessity. Suppose that x^* is the local minimum of (NLP). Then there exists some $\epsilon > 0$ such that $f(x^*) \le f(x)$ for all $x \in S(x^*;\epsilon) \cap X$. For every $(x,z) \in S(v^*;\epsilon) \cap \overline{X}$, we have $x \in S(x^*;\epsilon) \cap X$. Since $\bar{f}(x,z) = f(x)$ for all (x,z), there holds

$$\bar{f}(x^*, z^*) = f(x^*) \le f(x) = \bar{f}(x,z), \quad \text{for all } (x,z) \in S(v^*;\epsilon) \cap \overline{X}.$$

Conversely, suppose that for some $\epsilon > 0$, $\bar{f}(x^*, z^*) \le \bar{f}(x,z)$ for all $(x,z) \in S(v^*;\epsilon) \cap \overline{X}$. Since $g(x)$ is continuous, there exists some δ such that for all $x \in S(x^*;\delta) \cap \hat{X}$, we have $a(x) \in S(z^*;\epsilon/\sqrt{2})$.[4] Define $\hat{\epsilon} = \min\{\delta, \epsilon/\sqrt{2}\}$. Then for every $x \in S(x^*;\hat{\epsilon}) \cap X$, we have $\big(x, a(x)\big) \in S(v^*;\epsilon) \cap \overline{X}$. There holds

$$f(x^*) = \bar{f}\big(x^*, a(x^*)\big) \le \bar{f}\big(x, a(x)\big) = f(x), \quad \text{for all } x \in S(x^*;\hat{\epsilon}) \cap X.$$

Q.E.D.

For an example where [Lemma 1, P. 72] does not hold due to g being discountinuous, consider $f = x$ and $g(x) = -1$ for $x \ne 0$ and $g(x) = 0$ for $x = 0$. Then the feasible points of problem (38) are $(0,0)$ and $(x, \pm 1)$ for $x \ne 0$. Clearly, $(0,0)$ is a local minimum of problem (38), but $x = 0$ is not a local minimum of (NLP).

P. 74 Via transforming the inequality constraints to the equality ones, the scond order necessity condition states that

$$[y' \; v'] A \begin{bmatrix} y \\ v \end{bmatrix} \ge 0, \quad \text{for all } (y,v) \in \overline{Y}, \tag{39}$$

where the $(n+r)$ by $(n+r)$ matrix A is block diagonal and defined as

$$A = \begin{bmatrix} \nabla^2_{xx} L(x^*, \lambda^*, \mu^*) & & & \\ & 2\mu_1^* & & \\ & & \ddots & \\ & & & 2\mu_r^* \end{bmatrix}, \tag{40}$$

[4] To see this, we first note that $S(x^*;1/k) \cap \hat{X} \ne \emptyset$ for all k. Suppose there does not exist such a δ. Then for all k, there exists some $x_k \in S(x^*;1/k) \cap \hat{X}$, such that $|a(x_k) - z^*| \ge \epsilon/\sqrt{2}$. Since $x_k \to x^*$, we must have $a(x_k) \to z^*$ as g is continuous at x^*, which is a contradiction.

and the set $\overline{Y}$ is

$$\overline{Y} = \{(y, v) \mid \nabla h(x^*)'y = 0,\, \nabla g_j(x^*)'y + 2z_j^* v_j = 0,\, j = 1, \ldots, r\}. \tag{41}$$

We show that condition (39) implies that $y'\nabla^2_{xx}L(x^*, \lambda^*, \mu^*)y$ for all $y \in Y$, where

$$Y = \{y \mid \nabla h(x^*)'y = 0,\, \nabla g_j(x^*)'y = 0, \text{ for all } j \in A(x^*)\}.$$

Indeed, we have $z_j^* = 0$ for all $j \in A(x^*)$, and $z_j^* \neq 0$, $\mu_j^* = 0$ for all $j \notin A(x^*)$. For every $y \in Y$, we define v such that $v_j = 0$ for all $j \in A(x^*)$, and $v_j = -\nabla g_j(x^*)'y/(2z_j^*)$ for all $j \notin A(x^*)$, as $z_j^* \neq 0$. Then $(y, z) \in \overline{Y}$, which implies $[y'\ v']A[y'\ v']' \geq 0$. However, we also have $y'\nabla^2_{xx}L(x^*, \lambda^*, \mu^*)y = [y'\ v']A[y'\ v']'$, which is the desired inequality,

We next show that condition (39) implies $\mu_j^* \geq 0$ for all $j \in A(x^*)$. For every $j \in A(x^*)$, we set $y = 0$ and $v_i = 0$ for $i \neq j$ and $v_j = 1$. Then we have that $(y, v) \in \overline{Y}$. Moreover, $[y'\ v']A[y'\ v']' = 2\mu_j^*$. Therefore, $\mu_j^* \geq 0$.

P. 74 Let $(y, v) \neq 0$ such that $(y, v) \in \overline{Y}$, where $\overline{Y}$ is defined in Eq. (41). As discussed earlier in [P. 74], $v_j = -\nabla g_j(x^*)'y/(2z_j^*)$ for all $j \notin A(x^*)$. If $y \neq 0$, we have $[y'\ v']A[y'\ v']' \geq y'\nabla^2_{xx}L(x^*, \lambda^*, \mu^*)y > 0$. Otherwise, if $y = 0$, we have $v_j = 0$ for all $j \notin A(x^*)$. Since $(y, v) \neq 0$, there exists some $k \in A(x^*)$ such that $v_k \neq 0$. Therefore, $[y'\ v']A[y'\ v']' = \sum_{j \in A(x^*)} 2\mu_j^* v_j^2 \geq 2\mu_k^* v_k^2 > 0$.

P. 76

Lemma (1, P. 76)**.** *Let $x^* \geq 0$ be a feasible vector in (SCP). Then the following three conditions are equivalent:*

(a) For all $x \geq 0$, there holds $\nabla f(x^*)'(x - x^*) \geq 0$.

(b) We have $\partial f(x^*)/\partial x^i = 0$ if $x^{*i} > 0$, $i = 1, \ldots, n$, and $\partial f(x^*)/\partial x^i \geq 0$ if $x^{*i} = 0$, $i = 1, \ldots, n$.

(c) We have $x^* = [x^* - \alpha \nabla f(x^*)]^+$ for all $\alpha \geq 0$.

Proof. We first show the equivalence between (a) and (b). The sufficiency from (a) to (b) can be seen directly from the relations $\nabla f(x^*)'(x - x^*) = \sum_{i=1}^n \partial f(x^*)/\partial x^i (x^i - x^{*i}) \geq 0$. To show the necessity, for every i such that $x^{*i} = 0$, consider the feasible vector x such that $x^i = 1$ and $x^j = x^{*j}$ for all $j \neq i$. We obtain $\nabla f(x^*)'(x - x^*) = \partial f(x^*)/\partial x^i \geq 0$. For every i such that $x^{*i} > 0$, if $\partial f(x^*)/\partial x^i \neq 0$, we can find a feasible vector x such that $\nabla f(x^*)'(x - x^*) < 0$. Such a vector can be obtained by setting $x^j = x^{*j}$ for all $j \neq i$ and $x^i - x^{*i}$ takes the opposite sign of $\partial f(x^*)/\partial x^i$.

Then we show the equivalence between (b) and (c). The sufficiency from (b) to (c) is clear. As for the necessity, since $x^{*i} = \max\{x^{*i} - \alpha \partial f(x^*)/\partial x^{*i}, 0\}$, then if $x^{*i} = 0$, we must have $\partial f(x^*)/\partial x^{*i} \geq 0$. If $x^{*i} > 0$, we must have $\partial f(x^*)/\partial x^{*i} = 0$. This is exactly part (b). Q.E.D.

The equivalence between parts (b) and (c) will also be proved by using [COLMM Prop. 1.35, P. 78].

P. 79 The set I_1 should be interpreted as

$$I_1 = \{ i \,|\, x^i > 0 \text{ or } [x^i = 0 \text{ and } p^i < 0],\ i = 1, \ldots, r \}.$$

P. 80 Suppose that $p^=0$ for $i = 1, \ldots, r$. In view of [COLMM Eq. (18), P. 79], we have $x = x(\alpha)$ for all $\alpha \geq 0$, which is a contradiction.

P. 80 From the discussion here, we have $\nabla f(x)'\bar{p} > 0$ and $x(\alpha) = x - \alpha\bar{p}$. Apply [Lemma 3, P. 20] with $d = -\bar{p}$ and γ be any scalar in $(0, 1)$, we obtain the desired relation.

P. 82 We summarize the discussion here as some lemmas below.

Lemma (1, P. 82)**.** *Let $x_k \geq 0$ be a feasible vector in (SCP).*

(a) For all scalars $\alpha \geq 0$, there holds

$$\alpha \sum_{i \notin I_k^+} \frac{\partial f(x_k)}{\partial x^i} p_k^i + \sum_{i \in I_k^+} \frac{\partial f(x_k)}{\partial x^i} [x_k^i - x_k^i(\alpha)] \geq 0. \tag{42}$$

(b) The strict inequality in (42) *holds for all $\alpha > 0$ if and only if x_k is not a critical point.*

(c) The equality in (42) *holds for all $\alpha > 0$ if and only if x_k is a critical point.*

Proof. Without loss of generality, we assume that for some integer ℓ, we have $I_k^+ = \{\ell + 1, \ldots, n\}$. Accordingly, we can write D as the block diagonal matrix

$$D = \begin{bmatrix} \hat{D} & & & \\ & d^{\ell+1} & & \\ & & \ddots & \\ & & & d^n \end{bmatrix},$$

where $\hat{D}$ is positive definite.

(a) In view of the definition of p_k^i and the fact that $\hat{D}$ is positive definite, we have

$$\sum_{i \notin I_k^+} \frac{\partial f(x_k)}{\partial x^i} p_k^i = \left[\frac{\partial f(x_k)}{\partial x_k^1} \ \cdots \ \frac{\partial f(x_k)}{\partial x_k^\ell} \right] \hat{D} \left[\frac{\partial f(x_k)}{\partial x_k^1} \ \cdots \ \frac{\partial f(x_k)}{\partial x_k^\ell} \right]' \geq 0. \tag{43}$$

For every $i \in I_k^+$, we have $p^i > 0$. Therefore, $x_k^i - x_k^i(\alpha) = x_k^i - \max\{x_k^i - \alpha p_k^i, 0\} \geq 0$. Therefore, the second term in (42) is also nonnegative. Therefore, we obtain the desired inequality.

(b) To show necessity, let us assume that x_k is a critical point. In this case, $\partial f(x_k)/\partial x^i > 0$ implies that $x^i = 0$. Then we have $I_k^+ = I^+(x_k)$. Therefore, for every $i \in I_k^+$, $x_k^i - x_k^i(\alpha) = 0$ for all $\alpha > 0$. Moreover, $\partial f(x_k)/\partial x^i = 0$ for all $i \notin I_k^+$, as x_k is a critical point. Therefore, the first term in (42) is also 0 for all $\alpha > 0$.

To show sufficiency, suppose that x_k is not a critical point and there exists some $\alpha > 0$ such that the equality in Eq. (42) holds. Then for all $i \in I_k^+$, we have $p_k^i > 0$ and $x_k^i - \max\{x_k^i - \alpha p_k^i, 0\} = 0$. This means that $x_k^i = 0$ for all $i \in I_k^+$. Moreover, in view of Eq. (43), $\sum_{i \notin I_k^+} \partial f(x_k)/\partial x^i p_k^i = 0$ and the positive definiteness of $\hat{D}$ implies that $\partial f(x_k)/\partial x^i = 0$ for all $i \notin I_k^+$. This means that x_k is a critical point, which is a contradiction.

(c) The sufficiency part has been shown in the necessity proof of part (b). The necessity part can be shown by using the sufficiency part of the proof for part (b). Q.E.D.

Lemma (2, P. 82)**.** *Let $x_k \geq 0$ be a feasible vector in (SCP) and $\sigma \in (0,1)$ be some scalar. There exists some $\bar{\alpha} > 0$ such that*

$$f(x_k) - f\big[x_k(\alpha)\big] \geq \sigma \left\{ \alpha \sum_{i \notin I_k^+} \frac{\partial f(x_k)}{\partial x^i} p_k^i + \sum_{i \in I_k^+} \frac{\partial f(x_k)}{\partial x^i} [x_k^i - x_k^i(\alpha)] \right\} \tag{44}$$

for all $\alpha \in [0, \bar{\alpha}]$.

Proof. We continue to use the notation introduced in the proof of [Lemma 1, P. 82]. If x_k is a critical point, we have $x_k = x_k(\alpha)$ for all $\alpha \geq 0$ since D is also diagonal with respect to $I^+(x_k)$ and [COLMM Prop. 1.35, P. 78] applies. Therefore, the left hand side of Eq. (44) is zero for all $\alpha \geq 0$. Moreover, we have $I_k^+ = I^+(x_k)$ and $\partial f(x_k)/\partial x^i = 0$ for all $i \notin I_k^+$, as x_k is a critical point. Therefore, the right-hand-side of Eq. (44) is also zero for all $\alpha \geq 0$.

Suppose that x_k is not a critical point. The relation (44) holds for $\alpha = 0$ as both sides are zero. What remains is the case where $\alpha > 0$. In view that $I^+(x_k) \subset I_k^+$, we can assume, without loss of generality, that for some $r \geq \ell$, we have $I^+(x_k) = \{r+1, \ldots, n\}$. The proof arguments of part (b) in [COLMM P. 79] also apply here as D is diagonal with respect to $I^+(x_k)$. In particular, for the vector $\bar{p}$ defined in [COLMM P. 79] with x_k^i and p_k^i in place of x^i and p^i, respectively, there exists some $\alpha_1 > 0$ such that $x_k(\alpha) = x_k - \alpha \bar{p}$ for all $\alpha \in (0, \alpha_1)$. Moreover, $\nabla f(x_k)' \bar{p} > 0$. Therefore, applying [Lemma 3, P. 20] with $d = -\bar{p}$ and $\gamma = \sigma$, we have that for some $\bar{\alpha} \in (0, \alpha_1)$,

$$\begin{aligned} f(x_k) - f\big[x_k(\alpha)\big] \geq & \sigma \alpha \nabla f(x_k)' \bar{p} \\ = & \sigma \left\{ \alpha \sum_{i \notin I_k^+} \frac{\partial f(x_k)}{\partial x^i} \bar{p}^i + \sum_{i \in I_k^+} \frac{\partial f(x_k)}{\partial x^i} [x_k^i - x_k^i(\alpha)] \right\} \end{aligned} \tag{45}$$

for all $\alpha \in (0, \bar{\alpha}]$. We will show that

$$\sum_{i \notin I_k^+} \frac{\partial f(x_k)}{\partial x^i} \bar{p}^i \geq \sum_{i \notin I_k^+} \frac{\partial f(x_k)}{\partial x^i} p_k^i, \tag{46}$$

which together with Eq. (45), implies the desired relation (44) for $\alpha \in (0, \bar{\alpha}]$.

To this end, consider the sets

$$\hat{I}_1 = \{i \,|\, x_k^i > 0 \text{ or } [x_k^i = 0 \text{ and } p_k^i < 0],\ i = 1, \ldots, \ell\},$$
$$\hat{I}_2 = \{i \,|\, x_k^i = 0 \text{ and } p_k^i \geq 0,\ i = 1, \ldots, \ell\}.$$

From the definition of $\bar{p}$, it can be seen that $\bar{p}^i = p_k^i$ for $i \in \hat{I}_1$ and $\bar{p}^i = 0$ for $i \in \hat{I}_2$.[5] Moreover, for $i \in \hat{I}_2$, we have $\partial f(x_k)/\partial x^i \leq 0$. Therefore, for all $i \in \hat{I}_2$, $\partial f(x_k)/\partial x^i p_k^i \leq 0$. This yields

$$\begin{aligned} \sum_{i \notin I_k^+} \frac{\partial f(x_k)}{\partial x^i} p_k^i &= \sum_{i \in \hat{I}_1} \frac{\partial f(x_k)}{\partial x^i} p_k^i + \sum_{i \in \hat{I}_2} \frac{\partial f(x_k)}{\partial x^i} p_k^i \\ &\leq \sum_{i \in \hat{I}_1} \frac{\partial f(x_k)}{\partial x^i} \bar{p}^i = \sum_{i \notin I_k^+} \frac{\partial f(x_k)}{\partial x^i} \bar{p}^i. \end{aligned}$$

This concludes the proof. Q.E.D.

P. 83

Lemma (1, P. 83)**.** *Let $f : \Re^n \mapsto \Re$ be a twice continuously differentiable on an open set $S \subset \Re^n$. Then for every compact and convex set $C \subset S$, there exists some scalar L such that*

$$|\nabla f(x) - \nabla f(y)| \leq L|x - y| \quad \textit{for all } x, y \in C.$$

Proof. Applying first-order Taylor expansion to vector-valued function (see [P. 29]), we have

$$f(x) - f(y) = \int_0^1 \nabla^2 f\big[y + t(x - y)\big](x - y)dt.$$

Clearly, we have

$$|f(x) - f(y)|^2 = \bigg(\int_0^1 g(t)dt\bigg)'\bigg(\int_0^1 g(t)dt\bigg),$$

where

$$g(t) = \nabla^2 f\big[y + t(x - y)\big](x - y).$$

[5] In fact, we have $\hat{I}_2 = I_2$, where $I_2 = \{i \,|\, x_k^i = 0 \text{ and } p_k^i \geq 0,\ i = 1, \ldots, r\}$. Indeed, we clearly have $\hat{I}_2 \subset I_2$. Conversely, if $i \in I_2$, then $x_k^i = 0$ and $i \notin I^+(x_k)$, which implies that $\partial f(x)/\partial x^i \leq 0$, and consequently $i \notin I_k^+$. Therefore, $i \in \hat{I}_2$.

Applying [Lemma 3, P. 30], we have

$$\begin{aligned} |f(x) - f(y)|^2 &\leq \int_0^1 g(t)' g(t) dt \\ &= \int_0^1 (x-y)' \big(\nabla^2 f[y + t(x-y)]\big)^2 (x-y) dt. \end{aligned}$$

Consider the function $\tilde{g} : C \times Z \to \Re$ defined as $\tilde{g}(x, z) = z' \big(\nabla^2 f(x)\big)^2 z$, where $Z = \{z \mid |z| = 1\}$, and the function $h : C \mapsto \Re$ defined as $h(x) = \max_{z \in Z} \tilde{g}(x, z)$. Applying [Lemma 1, P. 30] with $\tilde{g}$ in place of g, we have h being continuous. As a result, the scalar L is finite, where $L^2 = \max_{x \in C} h(x)$. Since C is convex and $x, y \in C$, we have $y + t(x-y) \in C$ for $t \in [0, 1]$. This yields

$$|f(x) - f(y)|^2 \leq \int_0^1 L^2 |x-y|^2 dt = L^2 |x-y|^2.$$

Q.E.D.

P. 84 In view that the subsequence $\{x_k\}_K$ converges to $\bar{x}$, we have the monotone subsequence $\{f(x_k)\}_K$ converging to $f(\bar{x})$, as f is continuous. Then applying [Lemma 2, P. 25], we have $f(x_k) \to f(\bar{x})$.

P. 84 If $x_{\bar{k}}$ is a critical point for some $\bar{k}$, then $x_k = x_{\bar{k}}$ for all $k \geq \bar{k}$. Since $\bar{x}$ is assumed to be not critical, then x_k is not critical for all k. As a result, $w_k > 0$ for all k. Since the function $g : \Re^n \to \Re$ defined as $g(x) = \big|x - [x - M \nabla f(x)]^+\big|$ is continuous, and $g(\bar{x}) > 0$, then the sequence $\{w_k\}_K$ is bounded away from zero and bounded above as well. Then [COLMM Eq. (31), P. 83] implies that for all $i \in I_k^+$, the diagonal element of D_k that corresponds to i must be bounded below from zero and and bounded above for all $k \in K$. [COLMM Eq. (36), P. 84] holds by applying [COLMM Eq. (31), P. 83] with $z = z_k$ such that $z_k^i = \partial f(x_k)/\partial x^i$ for all $i \notin I_k^+$ and $z_k^i = 0$ for all $i \in I_k^+$. Here, we have $\bar{\lambda}_1 = \lambda_1 \inf\{w_k^{q_1}\}_K$ and $\bar{\lambda}_2 = \lambda_2 \sup\{w_k^{q_2}\}_K$. In other words, the eigenvalues of $\{D_k\}_K$ is lower-bounded by $\bar{\lambda}_1$ and upper-bounded by $\bar{\lambda}_2$.

P. 84 Let $\overline{K} \subset K$ be the infinite set such that $i \notin I_k^+$ for all $k \in \overline{K}$. Since $\lim_{k \to \infty, k \in \overline{K}} x_k^i = \bar{x}^i$, and $\partial f(\bar{x})/\partial x^i \neq 0$, then there exists some $\bar{k} \in \overline{K}$ and $\delta > 0$ such that $|\partial f(x_k)/\partial x^i| \geq \delta$ for all $k \geq \bar{k}$ and $k \in \overline{K}$. Consequently, by [COLMM Eq. (36), P. 84], we have

$$\sum_{j \notin I_k^+} p_k^j \frac{\partial f(x_k)}{\partial x^j} \geq \bar{\lambda}_1 \sum_{j \notin I_k^+} \left| \frac{\partial f(x_k)}{\partial x^j} \right|^2 \geq \bar{\lambda}_1 \left| \frac{\partial f(x_k)}{\partial x^i} \right|^2 \geq \bar{\lambda}_1 \delta^2 > 0.$$

Due to [COLMM Eq. (33), P. 84], we must have $\lim_{k \to \infty, k \in \overline{K}} \alpha_k \bar{\lambda}_1 \delta^2 = 0$, which implies $\{\alpha_k\}_{\overline{K}}$ converges to 0. In view of the following lemma with $a_k = \alpha_k$ and $a = 0$, we get [COLMM Eq. (37), P. 84].

Lemma (1, P. 84)**.** *Let $\{a_k\}$ be a real sequence such that $a_k \geq a$ for all k. Suppose it has a convergent subsequence $\{a_k\}_K$ whose limit is also a. Then we have for all k, $\inf_{n \geq k} a_n = a$, and consequently, $\liminf_{k \to \infty} a_k = a$.*

Proof. It suffices to show that for all k, $\inf_{n \geq k} a_n = a$. Since $a_k \geq a$ for all k, $\inf_{n \geq k} a_n \geq a$. Fix some k. Since the subsequence $\{a_i\}_K$ converges to a, then for every $\epsilon > 0$, there exits some $\bar{k} \in \overline{K}$ and $\bar{k} > k$ such that $a_{\bar{k}} < a + \epsilon$. As a result, $\inf_{n \geq k} a_n < a + \epsilon$ for all $\epsilon > 0$. Therefore, $\inf_{n \geq k} a_n \leq \inf\{a + \epsilon \,|\, \epsilon > 0\} = a$. Q.E.D.

P. 85 Let $\overline{K} \subset K$ be the set such that $i \in I_k^+$ for all $k \in \overline{K}$. For all $j \in I_k^+$, we have $\partial f(x_k)/\partial x^j > 0$ and $x_k^j - x_k^j(\alpha_k) \geq 0$. Since x_k^i converges to $\bar{x}^i > 0$ and $\partial f(\bar{x})/\partial x^i > 0$, there exists some integer $\bar{k}$, $\delta_1 > 0$, and $\delta_2 > 0$ such that $x_k^i > \delta_1$ and $\partial f(x_k)/\partial x^i > \delta_2$ for all $k \geq \bar{k}$ and $k \in \overline{K}$. From [COLMM Eq. (34), P. 84], we have

$$\lim_{k \to \infty, k \in \overline{K}} \frac{\partial f(x_k)}{\partial x^i}[x_k^i - x_k^i(\alpha_k)] = 0$$

$$\frac{\partial f(x_k)}{\partial x^i}[x_k^i - x_k^i(\alpha_k)] \geq \delta_2[x_k^i - x_k^i(\alpha_k)] \geq 0 \text{ for all } k \geq \bar{k},\ k \in \overline{K},$$

which implies

$$\lim_{k \to \infty, k \in \overline{K}} [x_k^i - x_k^i(\alpha_k)] = 0. \tag{47}$$

Next, we claim that there exits some $\tilde{k} \in \overline{K}$ such that $x_k^i(\alpha_k) = x_k^i - \alpha_k p_k^i$ for all $k \geq \tilde{k}$ and $k \in \overline{K}$. Suppose otherwise. Then there exists a set $\tilde{K} \subset \overline{K}$ such that $x_k^i(\alpha_k) = 0$ for all $k \in \tilde{K}$. Then we have $x_k^i - x_k^i(\alpha_k) = 0$ for all $k \in \tilde{K}$. Since $x_k^i > \delta_1$ for all $k \geq \bar{k}$ and $k \in \overline{K}$, this contradicts with Eq. (47). Therefore, for all $k \geq \tilde{k}$ and $k \in \overline{K}$, we have $x_k^i - x_k^i(\alpha_k) = \alpha_k p_k^i$. From [COLMM Eq. (35), P. 84], we have $p_k^i > \bar{\lambda}_1 \delta_2 > 0$ for all $k \geq \bar{k}$ and $k \in \overline{K}$. This yields $\{\alpha_k\}_{\overline{K}}$ converges to 0. Applying [Lemma 1, P. 84], we get the desired inequality.

P. 85

Lemma (1, P. 85)**.** *Let $g : \Re^r \mapsto \Re$ be a continuous function. Suppose that $\{y_k\} \subset \Re^r$ is a bounded sequence. Then the sequence $\{z_k\}$, defined as $z_k = g(y_k)$, is also bounded.*

Proof. Since $\{y_k\}$ is bounded, there exists a compact set Y such that $y_k \in Y$ for all k. Since g is continuous, the scalars $\underline{z} = \min_{y \in Y} g(y)$ and $\overline{z} = \max_{y \in Y} g(y)$ are finite, and $\underline{z} \leq z_k \leq \overline{z}$. Q.E.D.

Since $\{x_k\}_K$ is convergent, it is bounded. Therefore, applying [Lemma 1, P. 85] with $|\nabla f(\cdot)|$ and x_k in places of g and y_k, respectively, we have $\{\nabla f(x_k)\}_K$

is bounded. In view of the definition of $\bar{\lambda}_2$, see [P. 84], and the relation $p_k = D_k \nabla f(x_k)$, we have

$$|p_k|^2 = |D_k \nabla f(x_k)|^2 \leq \bar{\lambda}_2^2 |\nabla f(x_k)|^2.$$

Therefore, $\{p_k\}$ is also bounded.

To see that $\{x_k(\alpha)\}_K$ is uniformly bounded, we note first that $|x_k(\alpha)| \leq |x_k - \alpha p_k|$. We then apply the following lemma with $y = (x, p)$, $c = \alpha$, $C = [0, 1]$, and $g(x, p, \alpha) = |x - \alpha p|$.

Lemma (2, P. 85)**.** *Let $g : \Re^r \times \Re^m \mapsto \Re$ be a continuous function, and $C \subset \Re^m$ be a compact set. Suppose that $\{y_k\} \subset \Re^r$ is a bounded sequence. Then the sequence $\{z_k\}$, defined as $z_k = \max_{c \in C} g(y_k, c)$, is also bounded.*

Proof. Since the sequence $\{y_k\}$ is bounded, there exists some compact set $Y \subset \Re^r$ such that $y_k \in Y$ for all k. Consider the functions $\underline{g} : C \mapsto \Re$ and $\overline{g} : C \mapsto \Re$ defined as $\underline{g}(c) = \min_{y \in Y} g(y, c)$ and $\overline{g}(c) = \max_{y \in Y} g(y, c)$, respectively. According to [Lemma 1, P. 30], both $\overline{g}$ and $\underline{g}$ are continuous in c. Therefore, the scalars $\underline{z} = \min_{c \in C} \underline{g}(c)$ and $\overline{z} = \max_{c \in C} \overline{g}(c)$ are finite. For all k, we have $\underline{g}(c) \leq g(y_k, c) \leq \overline{g}(c)$. Therefore, $\underline{z} \leq z_k \leq \overline{z}$ for all k, which means that $\{z_k\}$ is bounded. Q.E.D.

P. 86 For $\alpha \geq 0$, if $x_k^i \geq \alpha p_k^i$, we have $|x_k^i - x_k^i(\alpha)| = \alpha |p_k^i|$. Otherwise for $x_k^i < \alpha p_k^i$, since $x_k^i \geq 0$, we have $|x_k^i - x_k^i(\alpha)| = |x_k^i| = x_k^i < \alpha p_k^i = \alpha |p_k^i|$.

P. 86 Since the eigenvalues of $\{D_k\}_K$ is upper-bounded by $\bar{\lambda}_2$, see [P. 84], and the relation $p_k = D_k \nabla f(x_k)$, we have

$$\sum_{i \notin I_k^+} (p_k^i)^2 \leq \bar{\lambda}_2^2 \sum_{i \notin I_k^+} \left| \frac{\partial f(x_k)}{\partial x^i} \right|^2 \leq \bar{\lambda}_2^2 \bar{\lambda}_1 \sum_{i \notin I_k^+} \frac{\partial f(x_k)}{\partial x^i} p_k^i,$$

where the second inequality is due to the first inequality in [COLMM Eq. (36), P. 84].

P. 87 Let $\bar{\alpha} \in (0, 1]$. Suppose $\beta \in (0, 1)$ and integer m satisfy $\beta^{m-1} > \bar{\alpha}$ and $\beta^m \leq \bar{\alpha}$. Multiplying β on both sides of $\beta^{m-1} > \bar{\alpha}$, we obtain $\beta^m \in (\beta\bar{\alpha}, \bar{\alpha}]$.

P. 87

Lemma (1, P. 87)**.** *Let $M(x)$ be an m by m symmetric matrix that depends continuously on the vector $x \in \Re^n$. Suppose that $\bar{x} \in \Re^n$ is some fixed vector. Then the following two statements are equivalent:*

(a) There holds that $z' M(\bar{x}) z > 0$ for all z such that $|z| = 1$.

(b) There exists some positive scalars δ, m_1, and m_2 such that for all $x \in S(\bar{x}; \delta)$, $m_1 |z|^2 \leq z' M(x) z \leq m_2 |z|^2$ for all z such that $|z| = 1$.

Proof. It is obvious that (b) implies (a). To see that (a) implies (b), consider the function $g(x,z) = z'M(x)z$ and the function $\underline{h}(x) = \min_{z\in Z} g(x,z)$, where $Z = \{z \,|\, |z| = 1\}$. Clearly, we have $\underline{h}(\bar{x}) > 0$. Applying [Lemma 1, P. 30], we have that $\underline{h}(x)$ is continuous. There exists some $\delta > 0$ such that for some $m_1 > 0$, $\underline{h}(x) \geq m_1$ for all $x \in S(\bar{x};\delta)$. Similarly, we can show the relation $z'M(x)z \leq m_2|z|^2$ by considering the function $\bar{h}(x) = \max_{z\in Z} g(x,z)$. Q.E.D.

Let us describe the sufficient conditions stated in [COLMM Prop. 1.31, P. 74] for (SCP). Since $L(x,\mu) = f(x)-\mu'x$, then $\nabla_x L(x^*,\mu^*) = 0$, which is $\nabla f(x^*)-\mu^* = 0$. Moreover, $\mu_i^* \geq 0$ and $\mu_i^* x^{*i} = 0$ for $i = 1,\ldots,n$. Together with $\nabla f(x^*) - \mu^* = 0$, we have $\nabla f(x^*) \geq 0$. Without loss of generality, we assume that there exists some r such that $A(x^*) = \{r+1,\ldots,n\}$. Moreover, $\nabla^2_{xx} L(x^*,\mu^*) = \nabla^2 f(x^*)$ As a result, we write the matrix $\nabla^2 f(x)$ as

$$\nabla^2 f(x) = \begin{bmatrix} H_1(x) & H_2(x) \\ H_2(x)' & H_3(x) \end{bmatrix},$$

where $H_1(x)$ is r by r matrix, and $H_3(x)$ is $(n-r)$ by $(n-r)$ matrix, both depends continuously on x. Therefore, for every $z \neq 0$ such that $z^i = 0$ for $i \in A(x^*)$, we must have $z = (\bar{z},0)$ for some nonzero $\bar{z} \in \Re^r$. Applying [Lemma 1, P. 87] with x^*, $H(x)$, and $\bar{z}$ in places of $\bar{x}$, $M(x)$, and z, respectively, we see that the [COLMM Eq. (53), P. 87] is equivalent to the corresponding condition given in [COLMM Prop. 1.31, P. 74].

P. 88 Since f is twice continuously differentiable on $S(x^*;\delta)$, applying [Lemma 1, P. 83] with $C = \{x \,|\, |x-x^*| \leq \delta_1\}$, where $\delta_1 \in (0,\delta)$, we have for all $x, \bar{x} \in C$, there exists some L such that $|\nabla f(x) - \nabla f(\bar{x})| \leq L|x-\bar{x}|$.

P. 88 Let us introduce some functions, which we will use throughout the discussion on the proof of [COLMM Prop. 1.37, P. 88]. Define $g : \Re^n \mapsto \Re^n$ as $g(x) = x - \big[x - M\nabla f(x)\big]^+$, with its ith component $g_i(x)$ given as $g_i(x) = x^i - \big[x^i - m^i \partial f(x)/\partial x^i\big]^+$. Define $w : \Re^n \mapsto \Re$ as $w(x) = |g(x)|$. Clearly, both w and g are continuous.

For all $i \in A(x^*)$, we have $x^{*i} = 0$ and $\partial f(x^*)/\partial x^i > 0$. Then there exists some $\gamma_1, \gamma_2 > 0$ such that for all $i \in A(x^*)$,

$$\partial f(x^*)/\partial x^i > \gamma_1/m^i,$$
$$|\partial f(x)/\partial x^i - \partial f(x^*)/\partial x^i| < \partial f(x^*)/\partial x^i - \gamma_1/m^i, \quad \text{for all } x \in S(x^*;\gamma_2).$$

Let $\gamma_3 = \min\{\gamma_1,\gamma_2\}$, then for all $i \in A(x^*)$, we have $\partial f(x^*)/\partial x^i - \gamma_3/m^i \geq \partial f(x^*)/\partial x^i > \gamma_1/m^i$, and $S(x^*;\gamma_3) \subset S(x^*;\gamma_2)$. Therefore, for all $i \in A(x^*)$, there holds

$$\partial f(x^*)/\partial x^i > \gamma_3/m^i, \tag{48}$$
$$|\partial f(x)/\partial x^i - \partial f(x^*)/\partial x^i| < \partial f(x^*)/\partial x^i - \gamma_3/m^i, \quad \text{for all } x \in S(x^*;\gamma_3). \tag{49}$$

For all x belonging to $S(x^*;\gamma_3)$ and satisfying $x \geq 0$, and $i \in A(x^*)$, we have $x^i = |x^i - x^{*i}| \leq |x - x^*| < \gamma_3$. In view of Eq. (48) and (49), we have $\partial f(x)/\partial x^i > \gamma_3/m^i$. Therefore, $x^i - m^i \partial f(x)/\partial x^i < \gamma_3 - \gamma_3 < 0$..

In summary, for all x belonging to $S(x^*;\gamma_3)$ and satisfying $x \geq 0$, and $i \in A(x^*)$, we have $\left[x^i - m^i \partial f(x)/\partial x^i\right]^+ = 0$ and $\partial f(x)/\partial x^i > 0$.

P. 88 Consider the function $h : \Re^n \mapsto \Re^n$ with its ith component $h_i(x) = x^i - w(x)$. The function h is continuous, and $h_i(x^*) = x^{*i}$. Therefore, $h_i(x^*) > 0$ for all $i \notin A(x^*)$. Due to continuity of h, there exists some $\gamma_4 > 0$ such that $h_i(x) > 0$ [equivalently, $x^i > w(x)$] for all $x \in S(x^*;\gamma_4)$ and $i \notin A(x^*)$. From the preceding discussion, we see that for all x belonging to $S(x^*;\gamma_3)$ and satisfying $x \geq 0$, and $i \in A(x^*)$, we have $\left[x^i - m^i \partial f(x)/\partial x^i\right]^+ = 0$, then we have $x^i = x^i - \left[x^i - m^i \partial f(x)/\partial x^i\right]^+ \leq w(x)$ for all $x \in S(x^*;\gamma_3)$ and $x \geq 0$. Define $\delta_2 = \min\{\gamma_3, \gamma_4\}$. Then for all $x \in S(x^*;\delta_2)$ and satisfying $x \geq 0$, we have $x^i \leq w(x)$ and $\partial f(x)/\partial x^i > 0$ for all $i \in A(x^*)$, and $x^i > w(x)$ for all $i \notin A(x)$.

Define $\tilde{I}^+(x)$ as the set $\tilde{I}^+(x) = \{i \,|\, 0 \leq x \leq w(x),\ \partial f(x)/\partial x^i > 0\}$. Then the above discussion shows that $A(x^*) = \tilde{I}^+(x)$ for all x belong to $S(x^*;\delta_2)$ and satisfying $x \geq 0$.

P. 88 Since $x^{*i} > 0$ for all $i \notin A(x^*)$, Let $\bar{\epsilon} = \min_{i \notin A(x^*)} x^{*i}/3$ and define $\delta_3 = \min\{\delta_2, \min_{i \notin A(x^*)} x^{*i}/3\}$. Then for all $x \in S(x^*;\delta_3)$ and satisfying $x \geq 0$ and $i \notin A(x^*)$, we have $x^i > x^{*i} - \delta_3 \geq 2x^{*i}/3 > \bar{\epsilon}$.

P. 89 Define $D(x)$ as positive definite matrix such that it is diagonal with respect to the set $\tilde{I}^+(x)$ and satisfies

$$\lambda_1 [w(x)]^{q_1} |z|^2 \leq z' D(x) z \leq \lambda_2 [w(x)]^{q_2} |z|^2, \quad \text{for all } z \in \Re^n, \tag{50}$$

where λ_1 and λ_2 are some positive scalars, and q_1 and q_2 are some nonnegative integers. In addition, denote the ith diagonal element of $D(x)$ as $d^{ii}(x)$, and it satisfies

$$\bar{\lambda}_1 \leq d^{ii}(x) \quad \text{for all } i \in \tilde{I}^+(x).$$

In addition, we define $p : \Re^n \mapsto \Re^n$ as $p(x) = D(x)\nabla f(x)$, with its ith component denoted as $p^i(x)$. We also introduce the function $\alpha : \Re^n \mapsto [\bar{\alpha}, 1]$. Finally, we introduce the function $\hat{x} : \Re^n \mapsto \Re^n$ defined as $\hat{x}(x) = [x - \alpha(x)p(x)]^+$, with its ith element denoted as $\hat{x}^i(x)$.

Next, we argue that for some $\gamma_5 \in (0, \delta_3]$, we have $x^i - \alpha(x)p^i(x) < 0$ for all $x \in S(x^*;\gamma_5)$ with $x \geq 0$, and $i \in A(x^*)$, which implies that

$$\hat{x}^i(x) = 0 \quad \text{for all } x \in S(x^*;\gamma_5) \text{ with } x \geq 0, \text{ and } i \in A(x^*). \tag{51}$$

We note first that in view of the discussion in [P. 88] and $\gamma_5 \leq \gamma_3$, we have $\partial f(x)/\partial x^i > 0$ for all $i \in A(x^*)$. As a result, $x^i - \alpha(x)p^i(x) \leq x^i - \bar{\alpha}\bar{\lambda}_1 \partial f(x)/\partial x^i$. Using the same arguments as those in [P. 88] with $\bar{\alpha}\bar{\lambda}_1$ in place of m^i, we can show that there exists some γ_5 such that for all $x \in S(x^*;\gamma_5)$ and $x \geq 0$, there holds $x^i - \alpha(x)p^i(x) < 0$ for all $i \in A(x^*)$.

In view of Eq. (50), we have

$$\sum_{i \notin A(x^*)} |p^i(x)|^2 \leq \lambda \sum_{i \notin A(x^*)} \left| \frac{\partial f(x)}{\partial x^i} \right|^2 \quad \text{for all } x \in S(x^*;\delta_3),\ x \geq 0, \tag{52}$$

where $\lambda = \max_{|x-x^*|\leq\delta_3, x\geq 0} \lambda_2[w(x)]^{q_2}$; cf. Eq. (50). Since $\partial f(x^*)/\partial x^i = 0$ for all $i \notin A(x^*)$, then there exists some $\gamma_6 \in (0,\delta_5)$ such that the right hand side of the above expression is strictly less than $\bar{\epsilon}^2/4$, where recall that $x^i > \bar{\epsilon}$ for all $i \notin A(x^*)$ for all $x \in S(x^*;\delta_3)$; cf [P. 88]. As a result, for all $i \notin A(x^*)$, $x^i - \alpha(x)p^i(x) > \bar{\epsilon} - \bar{\epsilon}/2 > 0$. Equivalently,

$$\hat{x}^i(x) > 0 \quad \text{for all } x \in S(x^*;\gamma_6) \text{ with } x \geq 0, \text{ and } i \notin A(x^*). \tag{53}$$

Combining Eqs. (51) and (53), we have that

$$A(x^*) = A\big(\hat{x}(x)\big) \quad \text{for all } x \in S(x^*;\gamma_6) \text{ with } x \geq 0. \tag{54}$$

Finally, there exists some $\gamma_7 \in (0,\gamma_6]$ such that the right hand side of Eq. (52) is no more than $\delta_3^2/8$ for all x with $|x - x^*| < \gamma_7$ and $x \geq 0$. Define $\delta_4 = \min\{\gamma_7, \delta_3/(2\sqrt{2})\}$. Then $|\hat{x}(x) - x^*| \leq |x - x^*| + |\hat{x}(x) - x|$. Clearly, for all x such that $|x - x^*| < \delta_4$ and $x \geq 0$, we have $\hat{x}^i(x) = 0$ for all $i \in A(x^*)$. Therefore, for all $x \in S(x^*;\delta_4)$ with $x \geq 0$, there holds

$$\begin{aligned}
|\hat{x}(x) - x|^2 &= \sum_{i \notin A(x^*)} |\hat{x}^i(x) - x^i|^2 + \sum_{i \in A(x^*)} |x^{*i} - x^i|^2 \\
&\leq \sum_{i \notin A(x^*)} |\hat{x}^i(x) - x^i|^2 + |x - x^*|^2 \\
&\leq \sum_{i \notin A(x^*)} |p^i(x)|^2 + |x - x^*|^2 = \delta_3^2/4,
\end{aligned}$$

where the last inequality is due to that $\alpha(x) \leq 1$. Therefore, for all $x \in S(x^*;\delta_4)$ with $x \geq 0$, there holds

$$|\hat{x}(x) - x^*| \leq |\hat{x}(x) - x| + |x - x^*| \leq \delta_3/2 + \delta_3/(2\sqrt{2}) \leq \delta_3.$$

Together with Eq. (54), we have established that for all $x \in S(x^*;\delta_4)$ with $x \geq 0$,

$$A(x^*) = A\big(\hat{x}(x)\big), \quad \big|\hat{x}(x) - x^*\big| \leq \delta_3.$$

P. 89 For all $x \in S(x^*;\delta_4)$ with $x \geq 0$, let us define the function $\bar{p}$ as $\bar{p} = \big(x - \hat{x}(x)\big)/\alpha(x)$, with its ith component denoted as $\bar{p}^i(x)$. In other words, $\bar{p}(x)$

is the effective gradient that satisfies $\hat{x}(x) = x - \alpha(x)\bar{p}(x)$. In particular, from the discussion in [P. 89], we have that

$$\bar{p}^i(x) = \begin{cases} p^i(x) & i \notin A(x^*), \\ x^i/\alpha(x) & i \in A(x^*). \end{cases}$$

Since $A(x^*) = \tilde{I}^+(x)$, so $D(x)$ is diagonal with respect to $A(x^*)$. As a result, we have

$$\nabla f(x)'\bar{p}(x) = \nabla f(x)'\tilde{D}(x)\nabla f(x),$$

where $\tilde{D}(x)$ is identical to $D(x)$, except that for every $i \in A(x^*)$, its ith diagonal element, denoted as $\tilde{d}^{ii}(x)$, is given as $d^{ii}(x)\beta$, where $\beta = \bar{p}^i(x)/p^i(x) > 0$. As a result, $\tilde{D}(x)$ is still positive definite, and for x that is not critical, we have

$$\nabla f(x)'\bar{p}(x) > 0.$$

Suppose that for some $\bar{k}$, we have $x_{\bar{k}} \in S(x^*; \delta_4)$ with $x_{\bar{k}} \geq 0$, then there holds $\bar{p}^i(x_{\bar{k}}) \geq 0$ for all $i \in A(x^*)$, and for all $k \geq \bar{k} + 1$, we have $\bar{p}^i(x_k) = 0$ for all $i \in A(x^*)$