KTH, SCHOOL OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

Box Condition of Some Bouneded Set

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1 PROBLEM STATEMENT

The pages refer to [Ch2_Abstract_DP_2ND_EDITION_20190902] edition.

Lemma 1 (P. 87)**.** *Given processor index set I being finite, and $\{X_\ell\}_{\ell\in I}$ is a partition of X, then it holds that*

$$\prod_{\ell\in I}\mathscr{B}(X_\ell)=\mathscr{B}(X). \tag{1.1}$$

Lemma 2 (P. 87)**.** *Given policy set $\mathscr{M}$ and processor index set I both being finite, and $\{X_\ell\}_{\ell\in I}$ is a partition of X, then it holds that*

$$\prod_{\ell\in I}\mathscr{J}_{|X_\ell}(\alpha)=\bigcap_{\ell\in I}\mathscr{J}_\ell(\alpha) \tag{1.2}$$

where

$$\mathscr{J}_{|X_\ell}(\alpha)=\left\{J_\ell\in\mathscr{B}(X_\ell)\,\middle|\,\max_{\mu\in\mathscr{M}}\sup_{x\in X_\ell}\frac{|J_\ell(x)-J_\mu(x)|}{v(x)}\leq\alpha\right\}, \tag{1.3}$$

$$\mathscr{J}_\ell(\alpha)=\left\{J\in\mathscr{B}(X)\,\middle|\,\max_{\mu\in\mathscr{M}}\sup_{x\in X_\ell}\frac{|J(x)-J_\mu(x)|}{v(x)}\leq\alpha\right\}. \tag{1.4}$$

Lemma 3 (P. 87)**.** *Given policy set $\mathscr{M}$ and processor index set I both being finite, and $\{X_\ell\}_{\ell\in I}$ is a partition of X, and given $\mathscr{J}(\alpha)$ and $\mathscr{J}_\ell(\alpha)$ as*

$$\mathscr{J}(\alpha)=\left\{J\in\mathscr{B}(X)\,\middle|\,\max_{\mu\in\mathscr{M}}\|J-J_\mu\|\leq\alpha\right\},$$

$$\mathscr{J}_\ell(\alpha)=\left\{J\in\mathscr{B}(X)\,\middle|\,\max_{\mu\in\mathscr{M}}\sup_{x\in X_\ell}\frac{|J(x)-J_\mu(x)|}{v(x)}\leq\alpha\right\}.$$

Then it holds that

$$\mathscr{J}(\alpha) = \bigcap_{\ell \in I} \mathscr{J}_\ell(\alpha). \tag{1.5}$$

Theorem 1 (P. 87)**.** *Given policy set $\mathscr{M}$ and processor index set I both being finite, and $\{X_\ell\}_{\ell\in I}$ is a partition of X, and given $\mathscr{J}(\alpha)$ and $\mathscr{J}_{|X_\ell}(\alpha)$ as*

$$\mathscr{J}(\alpha) = \left\{ J \in \mathscr{B}(X) \,\middle|\, \max_{\mu\in\mathscr{M}} \|J - J_\mu\| \le \alpha \right\},$$

$$\mathscr{J}_{|X_\ell}(\alpha) = \left\{ J_\ell \in \mathscr{B}(X_\ell) \,\middle|\, \max_{\mu\in\mathscr{M}} \sup_{x\in X_\ell} \frac{|J_\ell(x) - J_\mu(x)|}{v(x)} \le \alpha \right\}.$$

Then it holds that

$$\mathscr{J}(\alpha) = \prod_{\ell \in I} \mathscr{J}_{|X_\ell}(\alpha). \tag{1.6}$$

2 Elaboration

Proof of Lemma 1. First we show that $\mathscr{B}(X) \subseteq \prod_{\ell\in I} \mathscr{B}(X_\ell)$. Given $J \in \mathscr{B}(X)$ and denoted as J_ℓ the restriction of J on X_ℓ. Then we have

$$\sup_{x\in X_\ell} \frac{|J_\ell(x)|}{v(x)} \le \sup_{x\in X} \frac{|J(x)|}{v(x)} = \|J\| < \infty, \forall \ell$$

where the first inequality is due to that the supremum of an upper bounded real set is no less than the supremum of its subset. Therefore, we have $J_\ell \in \mathscr{B}(X_\ell)$ $\forall \ell$ and consequently $J \in \prod_{\ell\in I} \mathscr{B}(X_\ell)$.

On the other hand, given $J \in \prod_{\ell\in I} \mathscr{B}(X_\ell)$, and denote as M_ℓ the bound of $J_\ell \in \mathscr{B}(X_\ell)$, then we have

$$\frac{|J(x)|}{v(x)} \le \sum_{\ell\in I} M_\ell \chi_{X_\ell}(x)$$

where $\chi_{X_\ell}(\cdot)$ is the indicator functions defined on X. Then take supremum on both sides of the equation, we have

$$\sup_{x\in X} \frac{|J(x)|}{v(x)} \le \sup_{x\in X} \sum_{\ell\in I} M_\ell \chi_{X_\ell}(x) = \sup_{\ell\in I}\{M_\ell\} = \max_{\ell\in I}\{M_\ell\} < \infty.$$

Note that I being finite is needed. Otherwise, the bound of $\prod_{\ell\in I} J_\ell$ is $\sup_{\ell\in I}\{M_\ell\}$, which may be ∞. □

Proof of Lemma 2. Note that we need to apply the result of [Lemma 1, P. 87] that

$$\prod_{\ell\in I} \mathscr{B}(X_\ell) = \mathscr{B}(X)$$

to establish that the underline sets $\prod_{\ell\in I} \mathscr{B}(X_\ell)$ and $\mathscr{B}(X)$ are the same. With that fact in mind, we first show that $\prod_{\ell\in I} \mathscr{J}_{|X_\ell}(\alpha) \subseteq \bigcap_{\ell\in I} \mathscr{J}_\ell(\alpha)$. Indeed, for $J \in \prod_{\ell\in I} \mathscr{J}_{|X_\ell}(\alpha)$, it holds $\forall \ell \in I$, $\forall x \in X_\ell$, $\forall \mu \in \mathscr{M}$, that

$$\frac{|J(x) - J_\mu(x)|}{v(x)} = \frac{|J_\ell(x) - J_\mu(x)|}{v(x)} \le \max_{\mu\in\mathscr{M}} \sup_{x\in X_\ell} \frac{|J_\ell(x) - J_\mu(x)|}{v(x)} \le \alpha. \tag{2.1}$$

Take supremum over $x \in X_\ell$ on both sides and then take maximum over $\mu \in \mathscr{M}$ on both sides, and we have $J \in \bigcap_{\ell \in I} \mathscr{J}_\ell(\alpha)$.
On the other hand, given $J \in \bigcap_{\ell \in I} \mathscr{J}_\ell(\alpha)$, it holds $\forall \ell \in I$, $\forall x \in X_\ell$, $\forall \mu \in \mathscr{M}$, that

$$\frac{|J_\ell(x) - J_\mu(x)|}{v(x)} = \frac{|J(x) - J_\mu(x)|}{v(x)} \le \max_{\mu \in \mathscr{M}} \sup_{x \in X_\ell} \frac{|J(x) - J_\mu(x)|}{v(x)} \le \alpha, \tag{2.2}$$

which implies $J_\ell \in \mathscr{J}_{|X_\ell}(\alpha)$ $\forall \ell \in I$, and this concludes the proof. □

Proof of Lemma 3. Here we introduce two kinds of sets defined as

$$\mathscr{J}^\mu(\alpha) = \left\{ J \in \mathscr{B}(X) \,\middle|\, \|J - J_\mu\| \le \alpha \right\},$$
$$\mathscr{J}^\mu_\ell(\alpha) = \left\{ J \in \mathscr{B}(X) \,\middle|\, \sup_{x \in X_\ell} \frac{|J(x) - J_\mu(x)|}{v(x)} \le \alpha \right\}.$$

Then one can verify that between those two sets, it holds that

$$\mathscr{J}^\mu(\alpha) = \bigcap_{\ell \in I} \mathscr{J}^\mu_\ell(\alpha), \ \forall \mu \in \mathscr{M}.$$

In addition, one may verify that

$$\mathscr{J}(\alpha) = \bigcap_{\mu \in \mathscr{M}} \mathscr{J}^\mu(\alpha), \ \mathscr{J}_\ell(\alpha) = \bigcap_{\mu \in \mathscr{M}} \mathscr{J}^\mu_\ell(\alpha). \tag{2.3}$$

Therefore, we have

$$\begin{aligned} \mathscr{J}(\alpha) &= \bigcap_{\mu \in \mathscr{M}} \mathscr{J}^\mu(\alpha) \\ &= \bigcap_{\mu \in \mathscr{M}} \bigcap_{\ell \in I} \mathscr{J}^\mu_\ell(\alpha) \\ &= \bigcap_{\ell \in I} \bigcap_{\mu \in \mathscr{M}} \mathscr{J}^\mu_\ell(\alpha) \\ &= \bigcap_{\ell \in I} \mathscr{J}_\ell(\alpha) \end{aligned}$$

□

Proof of Theorem 1. Combining [Lemmas 1, 2, 3, P. 87], we get the desired result. □

Remark. *As noted in the proof of [Lemma 1, P. 87], it is needed to have I being finite in order to have the underline sets $\prod_{\ell \in I} \mathscr{B}(X_\ell)$ and $\mathscr{B}(X)$ being the same. However, as far as the proofs being concerned, it is not needed to have $\mathscr{M}$ being finite. The steps related in the proofs are* (2.1), (2.2), *and* (2.3), *which all holds if all* $\max_{\mu \in \mathscr{M}}$ *involved are replaced by* $\sup_{\mu \in \mathscr{M}}$ *properly.*

References

[1] Dimitri Bertsekas, *Abstract dynamic programming*, 2nd Edition, Athena Scientific, 2018.