KTH, SCHOOL OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

Sequence on the Extended Real Line

Yuchao Li

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1 PROBLEM STATEMENT

Lemma 1 (P. 42). *Given two extended real-valued sequences* $\{a_n\}$ *and* $\{b_n\}$ *where* $a_n, b_n \in \mathbb{R}^*$. *Assume that both* $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist in* $\mathbb{R}^*$. *Prove that*

$$a_n \le b_n \implies \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n.$$

Lemma 2 (P. 42). *Given* $\{a_n\}$ *with* $a_n \in \mathbb{R}$ *and* $a = \lim_{n\to\infty} a_n \in \mathbb{R}$, *and* $\{b_n\}$ *with* $b_n \in \mathbb{R}^*$ *and* $b = \lim_{n\to\infty} b_n \in \mathbb{R}^*$. *Prove that*

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Lemma 3 (P. 42). *Given* $\{a_n\} \in \mathbb{R}^*$ *which is monotone, prove that* $\{a_n\}$ *is convergent in* $\mathbb{R}^*$.

Lemma 4 (P. 42). *Given two real sequences* $\{a_n\}$ *and* $\{b_n\}$, *and assume that* $\limsup_{n\to\infty} a_n \in \mathbb{R}$, *show that*

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

2 ELABORATION

Proof of Lemma 1 Approach 1. If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist in $\mathbb{R}$, then denote their limits as a, b respectively. Then $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$, $|a_n - a| < \varepsilon$ and $|b_n - b| < \varepsilon$. Then the sequences $\{a_n\}_{n=N+1}^{\infty}$ and $\{b_n\}_{n=N+1}^{\infty}$ are real-valued and have the same limits a and b. Therefore, we have $a \le b$.

If $b = \lim_{n\to\infty} b_n$ exist in $\mathbb{R}$ but $\lim_{n\to\infty} a_n$ is not in $\mathbb{R}$, then given ε, $\exists N$ such that $\forall n > N$, $b_n < b + \varepsilon$. Since $a_n \le b_n$, then $\{a_n\}_{n=N+1}^{\infty}$ has an upper bound $b + \varepsilon < \infty$. Therefore, $\lim_{n\to\infty} a_n = -\infty$ and $-\infty < b$.
If $\lim_{n\to\infty} b_n = -\infty$, then $\forall x \in \mathbb{R}$, $\exists N$ such that $\forall n > N$, $b_n < x$. Since $a_n \le b_n$, we conclude $\lim_{n\to\infty} a_n = -\infty$.
If $\lim_{n\to\infty} b_n = \infty$, then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$ holds regardless the value of $\lim_{n\to\infty} a_n$. □

Proof of Lemma 1 Approach 2. Denote the limits as $a, b \in \mathbb{R}^*$ respectively. Apply the metric on $\mathbb{R}^*$ introduced in [Note 2]. Then we have $\lim_{n\to\infty} d(a_n, a) = 0$ and $\lim_{n\to\infty} d(b_n, b) = 0$. If $a > b$, then denote $2\varepsilon = d(a, b)$. Then $\exists N$ such that $\forall n > N$, $d(a_n, a) < \varepsilon$ and $d(b_n, b) < \varepsilon$, which implies $a_n > b_n$, which concludes the proof. □

Proof of Lemma 2. If $b \in \mathbb{R}$, then given $\varepsilon > 0$, $\exists N$ such that $|b_n - b| < \varepsilon$. So the sequence $\{b_n\}_{n=N+1}^{\infty}$ is real-valued and the equality follows.
If $b = \pm\infty$, take $b = \infty$ as an example. Then we have $\{b_n\}$ is unbounded above. Since $a \in \mathbb{R}$, then given $\varepsilon > 0$ $\exists N$ such that $\forall n > N$ $a_n > a - \varepsilon$, then we have sequence $\{a_n + b_n\}$ is unbounded above. Therefore the equality holds since $\lim_{n\to\infty}(a_n + b_n) = \infty = \infty + a = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$. □

Proof of Lemma 3. Take nonincreasing sequence as an example. If $\{a_n\} \cap \{-\infty\} \neq \emptyset$, then $\exists a_K = -\infty$. Since $\forall n > K$ $a_n \le a_K$, then the convergence follows.
If $\{a_n\} \cap \{-\infty\} = \emptyset$ but $\{a_n\} \cap \mathbb{R} \neq \emptyset$, then $\exists a_K \in \mathbb{R}$, which means $\{a_n\}_{n=K}^{\infty}$ is a real-valued sequence. Then it's either bounded below or unbounded below. Either case, the convergence follows.
If $\{a_n\} \cap \{-\infty\} = \emptyset$ and $\{a_n\} \cap \mathbb{R} = \emptyset$, then $a_n = \infty$. Then the convergence follows. □

Proof of Lemma 4. Given n, $\forall k \ge n$, $a_k \le \sup_{k\ge n} a_k$, and $b_k \le \sup_{k\ge n} b_k$, consequently $a_k + b_k \le \sup_{k\ge n} a_k + \sup_{k\ge n} b_k$. Therefore, we have

$$\sup_{k\ge n} \left(a_k + b_k\right) \le \sup_{k\ge n} a_k + \sup_{k\ge n} b_k.$$

Since both sides are monotonically nonincreasing, so by Lemma 3 both sides have limits in $\mathbb{R}^*$, and by Lemma 1, we have

$$\lim_{n\to\infty} \sup_{k\ge n}(a_k + b_k) \le \lim_{n\to\infty} (\sup_{k\ge n} a_k + \sup_{k\ge n} b_k) \tag{2.1}$$

In addition, by Lemma 3, we have $\{\sup_{k\ge n} a_k\}_{n=1}^{\infty}$ and $\{\sup_{k\ge n} b_k\}_{n=1}^{\infty}$ converge in $\mathbb{R}^*$. Since $\limsup_{n\to\infty} a_n \in \mathbb{R}$, denote $\bar{a} = \limsup_{n\to\infty} a_n$. Then given $\varepsilon > 0$, $\exists N$ such that $\sup_{k\ge N} a_k \in (\bar{a} - \varepsilon, \bar{a} + \varepsilon)$. Therefore, $\{\sup_{k\ge n} a_k\}_{n=1}^{\infty}$ is in in the interval $(\bar{a} - \varepsilon, S] \subset \mathbb{R}$ where

$$S = \max\{a_1, ..., a_N, \bar{a} + \varepsilon\}.$$

Therefore, $\{\sup_{k\ge n} a_k\}_{n=1}^{\infty}$ is real-valued sequence converging in $\mathbb{R}$. Then by Lemma 2, we have

$$\lim_{n\to\infty} (\sup_{k\ge n} a_k + \sup_{k\ge n} b_k) = \lim_{n\to\infty} \sup_{k\ge n} a_k + \lim_{n\to\infty} \sup_{k\ge n} b_k. \tag{2.2}$$

Combine Eqs. (2.1) and (2.2), we get the desired result. □

References

[1] Dimitri Bertsekas, *Abstract dynamic programming*, 2nd Edition, Athena Scientific, 2018.