KTH, SCHOOL OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

State-Dependent Weighted Multistep Mappings

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1 PROBLEM STATEMENT

Lemma 1 (P. 64). *Given a real valued sequence* $\{a_\ell\}$ *and assume that the sequence* $\{\sum_{\ell=1}^{n} a_\ell\}_{n=1}^{\infty}$ *converges with* $\sum_{\ell=1}^{\infty} a_\ell \in \mathbb{R}$. *It holds that*

$$|\sum_{\ell=1}^{\infty} a_\ell| \le \sum_{\ell=1}^{\infty} |a_\ell|. \tag{1.1}$$

Theorem 1 (P. 64). *Let the set of mappings* $T_\mu : \mathscr{B}(X) \to \mathscr{B}(X)$, $\mu \in \mathscr{M}$ *satisfy Assumption 2.1.2. Consider the mappings* $T_\mu^{(w)} : \mathscr{B}(X) \to \mathscr{R}(X)$ *defined by*

$$\big(T_\mu^{(w)} J\big)(x) = \sum_{\ell=1}^{\infty} w_\ell(x)\big(T_\mu^{\ell} J\big)(x),\ x \in X,\ J \in \mathscr{B}(X), \tag{1.2}$$

where $w_\ell(x)$ *are nonnegative scalars such that for all* $x \in X$,

$$\sum_{\ell=1}^{\infty} w_\ell(x) = 1.$$

Then the mapping $T_\mu^{(w)}$ *is well defined; namely for all* $x \in X$, $J \in \mathscr{B}(X)$, *the sequence*

$$\Big\{\sum_{\ell=1}^{n} w_\ell(x)\big(T_\mu^{\ell} J\big)(x)\Big\}_{n=1}^{\infty} \tag{1.3}$$

converges with a limit in $\mathbb{R}$.

Theorem 2 (P. 64). *Let the set of mappings* $T_\mu : \mathscr{B}(X) \to \mathscr{B}(X)$, $\mu \in \mathscr{M}$ *satisfy Assumption 2.1.2. Consider the mappings* $T_\mu^{(w)} : \mathscr{B}(X) \to \mathscr{R}(X)$ *defined in Eq.* (1.2). *It holds that* $T_\mu^{(w)}\mathscr{B}(X) \subset \mathscr{B}(X)$, *namely,* $T_\mu^{(w)} : \mathscr{B}(X) \to \mathscr{R}(X)$ *is in fact* $T_\mu^{(w)} : \mathscr{B}(X) \to \mathscr{B}(X)$; *and* $T_\mu^{(w)}$ *is a contraction.*

Theorem 3 (P. 64)**.** *Consider sequence* $\{T_\mu^{(\lambda_n)} J\}$ *defined by*

$$T_\mu^{(\lambda_n)} J = (1-\lambda) \sum_{\ell=1}^{n} \lambda^{\ell-1} T_\mu^{\ell} J.$$

The sequence $\{T_\mu^{(\lambda_n)} J\}$ *converges to some element* $T_\mu^{(\lambda_\infty)} J \in \mathscr{B}(X)$*. In addition, it coincides with the function* $T_\mu^{(\lambda)} J$ *defined by point-wise limit, viz.,* $T_\mu^{(\lambda_\infty)} J = T_\mu^{(\lambda)} J$*.*

2 Elaboration

Proof of Lemma 1. Due to continuity of $|\cdot|$, we have

$$\lim_{n\to\infty} |\sum_{\ell=1}^{n} a_\ell| = |\sum_{\ell=1}^{\infty} a_\ell|.$$

Since $\forall n$, it holds that

$$|\sum_{\ell=1}^{n} a_\ell| \le \sum_{\ell=1}^{n} |a_\ell|,$$

which is due to triangular inequality, then taking limits on both sides and we get the desired inequality. Note that the limit on right-hand side of Eq. (1.1) need not be checked as it's nonnegative and therefore monotone and it may be $+\infty$. □

Proof of Theorem 1. Since T_μ is a contraction, we have $\big(T_\mu^\ell J\big)(x) \to J_\mu(x) \in \mathbb{R}$, $\forall x \in X$. Therefore, $\{\big(T_\mu^\ell J\big)(x)\}_{\ell=1}^{\infty}$ is bounded. Denote the bound as $M_\mu(x) \in \mathbb{R}$. Then $\forall n$, it holds that

$$|\sum_{\ell=1}^{n} w_\ell(x)\big(T_\mu^\ell J\big)(x)| \le \sum_{\ell=1}^{n} w_\ell(x)|\big(T_\mu^\ell J\big)(x)| \le \sum_{\ell=1}^{n} w_\ell(x) M_\mu(x) \le M_\mu(x),$$

namely the sequence of (1.3) is bounded. If $J_\mu(x) > 0$, then $\exists N$ such that $\big(T_\mu^\ell J\big)(x) > 0$ $\forall \ell > N$. Therefore, $\{\sum_{\ell=1}^{n} w_\ell(x)\big(T_\mu^\ell J\big)(x)\}_{n=N}^{\infty}$ is monotonically nondecreasing and bounded by $M_\mu(x)$. Therefore the sequence (1.3) converges with the limit $\sum_{\ell=1}^{\infty} w_\ell(x)\big(T_\mu^\ell J\big)(x) \in \mathbb{R}$. If $J_\mu(x) < 0$, similar arguments applies. If $J_\mu(x) = 0$, then $\forall \varepsilon$, $\exists N$ such that $\forall \ell > N$, $|\big(T_\mu^\ell J\big)(x)| < \varepsilon$. Therefore, $\forall k$, it holds that

$$\begin{aligned}\Big|\sum_{\ell=1}^{N} w_\ell(x)\big(T_\mu^\ell J\big)(x) - \sum_{\ell=1}^{N+k} w_\ell(x)\big(T_\mu^\ell J\big)(x)\Big| &= \Big|\sum_{\ell=N+1}^{N+k} w_\ell(x)\big(T_\mu^\ell J\big)(x)\Big| \\ &\le \sum_{\ell=N+1}^{N+k} w_\ell(x)|\big(T_\mu^\ell J\big)(x)| \\ &\le \sum_{\ell=N+1}^{N+k} w_\ell(x)\varepsilon \\ &\le \varepsilon, \end{aligned} \tag{2.1}$$

which implies that the sequence (1.3) is Cauchy. [For details of Eq. (2.1) implying Cauchy, refer to Question 4, HW5, FEO3230.] As a result, sequence (1.3) converges in $\mathbb{R}$. Therefore, $\forall J \in \mathscr{B}(X)$, $x \in X$, sequence (1.3) converges in $\mathbb{R}$. Namely $T_\mu^{(w)} : \mathscr{B}(X) \to \mathscr{R}(X)$. □

Proof of Theorem 2. Due to Theorem 1, $\forall J \in \mathscr{B}(X)$ and $x \in X$, $\big(T_\mu^{(w)} J\big)(x)$ is well-defined and is a real value. In particular, for $J = J_\mu$, we have $T_\mu^{(w)} J_\mu = J_\mu$ (one may verify this equality by checking the definition Eq. (1.2)). Then we have

$$\begin{aligned}
|\big(T_\mu^{(w)} J\big)(x) - J_\mu(x)| &= \Big|\sum_{\ell=1}^{\infty} w_\ell(x)\big(T_\mu^\ell J\big)(x) - J_\mu(x)\Big| \\
&= \Big|\sum_{\ell=1}^{\infty} w_\ell(x)\big(T_\mu^\ell J\big)(x) - \sum_{\ell=1}^{\infty} w_\ell(x)\big(T_\mu^\ell J_\mu\big)(x)\Big| \\
&= \Big|\sum_{\ell=1}^{\infty} w_\ell(x)\Big(\big(T_\mu^\ell J\big)(x) - \big(T_\mu^\ell J_\mu\big)(x)\Big)\Big| \\
&\le \sum_{\ell=1}^{\infty} w_\ell(x)|\big(T_\mu^\ell J\big)(x) - \big(T_\mu^\ell J_\mu\big)(x)|
\end{aligned}$$

where the last inequality holds due to Lemma 1. Since T_μ is a contraction, $\forall \ell$, it holds that

$$|\big(T_\mu^\ell J\big)(x) - \big(T_\mu^\ell J_\mu\big)(x)| \le \alpha^\ell \|J - J_\mu\| v(x).$$

Therefore, we have

$$|\big(T_\mu^{(w)} J\big)(x) - J_\mu(x)| \le \sum_{\ell=1}^{\infty} w_\ell(x)\alpha^\ell \|J - J_\mu\| v(x) \le \bar{\alpha}\|J - J_\mu\| v(x) \tag{2.2}$$

where $\bar{\alpha}$ is given as

$$\bar{\alpha} = \sup_{x\in X} \sum_{\ell=1}^{\infty} w_\ell(x)\alpha^\ell \le \alpha.$$

Note that for all $x \in X$, the sequence $\big\{\sum_{\ell=1}^{n} w_\ell(x)\alpha^\ell\big\}_{n=1}^{\infty}$ converges in real since it's monotonically nondecreasing and upper bounded by α. Therefore $\bar{\alpha}$ is well-defined. Due to triangular inequality, from Eq. (2.2), we have

$$\frac{|\big(T_\mu^{(w)} J\big)(x)|}{v(x)} \le \bar{\alpha}\|J - J_\mu\| + \frac{|J_\mu(x)|}{v(x)}.$$

Take supremum over x on both sides and due to $J_\mu \in \mathscr{B}(X)$, we have $T_\mu^{(w)} J \in \mathscr{B}(X)$. Regarding the contraction proof, refer to Exercise 1.3, P. 38, [1] for details. □

Proof of Theorem 3. Since $\lim_{n\to\infty} \|T_\mu^n J - J_\mu\| = 0$, we have $\lim_{n\to\infty} \|T_\mu^n J\| = \|J_\mu\|$ [cf. Theorem P. 42]. Therefore $\{\|T_\mu^n J\|\}$ is bounded. Denote its bound as M_μ. Therefore, $\forall \varepsilon$, $\exists N$ such that

$\forall k$ it holds that

$$\begin{aligned}
\| T_\mu^{(\lambda_N)} J - T_\mu^{(\lambda_{N+k})} J \| &= \| (1-\lambda) \sum_{\ell=1}^{N} \lambda^{\ell-1} T_\mu^{\ell} J - (1-\lambda) \sum_{\ell=1}^{N+k} \lambda^{\ell-1} T_\mu^{\ell} J \| \\
&= \| (1-\lambda) \sum_{\ell=N+1}^{N+k} \lambda^{\ell-1} T_\mu^{\ell} J \| \\
&\le (1-\lambda) \sum_{\ell=N+1}^{N+k} \lambda^{\ell-1} \| T_\mu^{\ell} J \| \\
&\le (1-\lambda) \sum_{\ell=N+1}^{N+k} \lambda^{\ell-1} M_\mu \\
&\le \lambda^N M_\mu \\
&\le \varepsilon,
\end{aligned}$$

which implies $\{T_\mu^{(\lambda_n)} J\}$ is Cauchy. Since $\mathscr{B}(X)$ is complete, then it is also convergent. Denote its limit as $T_\mu^{(\lambda_\infty)} J$. Since convergence in norm implies point-wise convergence and limit in $\mathbb{R}$ is unique, then $\forall x \in X$, it holds that $\big(T_\mu^{(\lambda_\infty)} J\big)(x) = \big(T_\mu^{(\lambda)} J\big)(x)$. □

References

[1] Dimitri Bertsekas, *Abstract dynamic programming*, 2nd Edition, Athena Scientific, 2018.