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Norm Construction for Linear Mapping

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1 Problem statement

In the norm construction given in the footnote, P. 326, [1], show that β is independent of δ.

2 Elaboration

Denote the matrix $\hat{J} = \Lambda + B(\delta)$ where Λ is the diagonal matrix which contains the eigenvalues of $\hat{J}$ and $B(\delta)$ is a super-diagonal matrix whose super-diagonal elements are either δ or 0. Therefore, $\forall z \in \mathbb{R}^n$, it holds that

$$\begin{aligned} z'\hat{J}'\hat{J}z &= z'\big(\Lambda + B(\delta)\big)'\big(\Lambda + B(\delta)\big)z \\ &= z'\Lambda^2 z + z'\Lambda B(\delta)z + z'B(\delta)'\Lambda z + z'B(\delta)'B(\delta)z \\ &= z'\Lambda^2 z + 2z'\Lambda B(\delta)z + z'B(\delta)'B(\delta)z \\ &\le z'(\Lambda^2 + \delta^2 I)z + 2z'\Lambda B(\delta)z \end{aligned} \tag{2.1}$$

where I is the identity matrix and the inequality holds since $|z'B(\delta)'B(\delta)z| \le z'\delta^2 Iz$. Here we applys a property of superdiagonal matrix, for example, a straight forward computation shown that

$$\begin{bmatrix} 0 & \delta & 0 \\ 0 & 0 & \delta \\ 0 & 0 & 0 \end{bmatrix}' \begin{bmatrix} 0 & \delta & 0 \\ 0 & 0 & \delta \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta^2 & 0 \\ 0 & 0 & \delta^2 \end{bmatrix}.$$

Note that the components of $z'\Lambda B(\delta)z$ have the form $\delta\lambda_\ell z_\ell z_{\ell+1}$ where λ_ℓ is an eigenvalue of A (also of $\hat{J}$). Therefore, we have

$$|2\delta\lambda_\ell z_\ell z_{\ell+1}| \le \delta\sigma(A)(z_\ell^2 + z_{\ell+1}^2) \Rightarrow 2z'\Lambda B(\delta)z \le \sum_{\ell=1}^{n-1} |2\delta\lambda_\ell z_\ell z_{\ell+1}| < 2\delta\sigma(A)\|z\|^2$$

Note that the last inequality is strict. Then continue from Eq. (2.1), we have

$$
\begin{aligned}
z'\hat{J}'\hat{J}z &< z'(\Lambda^2 + \delta^2 I + 2\delta\sigma(A)I)z \\
&\le (\sigma(A)^2 + \delta^2 + 2\delta\sigma(A))z'Iz \\
&= (\sigma(A) + \delta)\|z\|^2.
\end{aligned}
$$

So one typical β is 1 and the bound is not tight.

References

[1] Dimitri Bertsekas, *Abstract dynamic programming*, 2nd Edition, Athena Scientific, 2018.