KTH, SCHOOL OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

Convergence of Double Sequences

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1 PROBLEM STATEMENT

For extended real line $\mathbb{R}^*$, we apply the metric given in [Note 2]. That is, $\forall x, y \in \mathbb{R}^*$, the metric is defined as

$$d(x,y) = |f(x) - f(y)| \tag{1.1}$$

where $f : \mathbb{R}^* \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is given by

$$f(x) = \begin{cases} -\frac{\pi}{2}, & x = -\infty \\ \arctan x, & x \in \mathbb{R}, \\ \frac{\pi}{2}, & x = \infty. \end{cases} \tag{1.2}$$

Then similar to the Definition 2.1 in [Note 1], we say a double sequence $\{s(n,m)\}$ that is defined on $\mathbb{R}^*$ converges to $a \in \mathbb{R}^*$, which we write as $\lim_{n,m\to\infty} s(n,m) = a$, if $\forall \varepsilon > 0$, $\exists N(\varepsilon)$, such that $\forall n, m \geq N(\varepsilon)$, it holds that

$$d\big(s(n,m), a\big) < \varepsilon. \tag{1.3}$$

Prove the following statement.
Let $\lim_{n,m\to\infty} s(n,m) = a \in \mathbb{R}^*$. Then $\lim_{n\to\infty}\big(\lim_{m\to\infty} s(n,m)\big) = a$ if and only if $\lim_{m\to\infty} s(n,m)$ exists for each $n \in \mathbb{N}$.

2 ELABORATION

1. For necessity, since $\lim_{n\to\infty}\big(\lim_{m\to\infty} s(n,m)\big)$ is well-defined, then $\lim_{m\to\infty} s(n,m)$ exists for each $n \in \mathbb{N}$.

2. For sufficiency, since we know that $\lim_{n,m\to\infty} s(n,m) = a \in \mathbb{R}^*$, then given any $\varepsilon > 0$, $\exists N_1(\varepsilon)$ such that $\forall n, m \geq N_1(\varepsilon)$, it holds that $d\big(s(n,m), a\big) < \varepsilon/2$. Since $\lim_{m\to\infty} s(n,m)$ exists for each $n \in \mathbb{N}$, we denote $\lim_{m\to\infty} s(n,m) = c_n$. For a given n, per definition, given any $\varepsilon > 0$, $\exists N_2(n,\varepsilon)$ such that $\forall m \geq N_2(n,\varepsilon)$, $d\big(s(n,m), c_n\big) < \varepsilon/2$.

   Therefore, given $\varepsilon > 0$, for every $n \geq N_1(\varepsilon)$, chose $m_n > \max\{N_1(\varepsilon), N_2(n,\varepsilon)\}$, it holds that
$$d(c_n, a) \leq d\big(c_n, s(n, m_n)\big) + d\big(s(n, m_n), a\big) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

   Namely, $d(c_n, a) < \varepsilon$ holds $\forall n \geq N_1(\varepsilon)$, which concludes the proof.